Modularity and Enumerative Geometry of Physical Invariants

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Abstract/Resumo

Abstract in English

In theoretical physics, more specifically in string theory, it is predicted that the so-called Calabi-Yau manifolds play an important role in the structure of our universe. Of importance in this physical theory are the enumerative invariants associated to such manifolds. Among these, we focus on the study of Gromov-Witten invariants (counts of stable maps) and Donaldson-Thomas invariants (counts of ideal sheaves). Our main contribution is, in the first part of this thesis, to introduce an algebraic framework for the computation of open string (or real) Gromov-Witten invariants for the quintic threefold (first achieved by Walcher). We show, using Hodge theory, that there is a possible modular interpretation to some generating series, based on previous work by Movasati for the closed string case. In the second part, we introduce, using the motivic invariants defined in the work of Kontsevich and Soibelman, new refinements of Donaldson-Thomas invariants. These are based on a version of enumerative geometry over arbitrary fields developed and studied by Kass-Wickelgren, Levine, and others, in which the counts obtained are not integers but quadratic forms. We also give examples and applications. We finish the thesis by posing some questions for further research and on possible relations between the two parts of this work.

Resumo em Português

Em Física teórica, mais especificamente em teoria das cordas, prevê-se que variedades de Calabi-Yau desempenhem um papel importante na estrutura de nosso universo. De importância nessa teoria física são os invariantes enumerativos associados a essas variedades. Entre eles, focamos no estudo dos invariantes de Gromov-Witten (contagens de mapas estáveis) e nos invariantes de Donaldson-Thomas (contagem de feixes de ideais). Nossa principal contribuição é, na primeira parte dessa Tese, introduzir um *framework* algébrico para o cálculo de invariantes de Gromov-Witten para cordas abertas, também chamados de invariantes reais, para a quíntica tridimensional (feito pela primeira vez por Walcher). Nós mostramos, usando teoria de Hodge, que existe uma possível interpretação modular para algumas séries geradoras, baseando-nos no trabalho anterior de Movasati para o caso da corda fechada. Na segunda parte, a partir dos invariantes motívicos definidos no trabalho de Kontsevich and Soibelman introduzimos, novos refinamentos de invariantes de Donaldson-Thomas. Estes são baseados numa versão de geometria enumerativa desenvolvida e estudada por Kass-Wickelgren, Levine e outros, na qual as contagens obtidas não são números inteiros, mas formas quadráticas. Também damos alguns exemplos e aplicações. Finalizamos a tese colocando algumas perguntas para pesquisa futura e sobre possíveis relações entre as duas partes deste trabalho.

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Contents

Introduction

This work is divided into two parts: the first one on modularity of open Gromov-Witten invariants in the framework of the *Gauss-Manin Connection in Disguise* (GMCD) program, and the second one on arithmetic and motivic refinements of Donaldson-Thomas invariants. The first part of the work was developed at IMPA during the academic year 2021-2022 and is summarized in the paper [23] and the second part was developed at the University of Heidelberg during 2022-2023, summarized in the paper [24].

Modularity of Open Gromov-Witten Invariants

Quasimodular forms for $SL(2,\mathbb{Z})$ are holomorphic functions on the upper half-plane which can be given by polynomials in the Eisenstein series E_2, E_4, E_6, defined as below.

$$E_{2k} := 1 - \frac{4k}{B_{2k}} \sum_{d=0}^{\infty} \sigma_{2k-1}(d) q^d$$

where $q = e^{2\pi i z}$ is a coordinate on the unit disk and z is a coordinate on the upper half-plane. Also, B_{2k} are Bernoulli numbers and $\sigma_{2k-1}(d)$ is the sum of the $(2k-1)$th powers of divisors of d.

Quasimodular forms (in particular modular forms, which are quasimodular forms that do not involve E_2) have many applications in number theory and physics, and knowing whether a function has a modular behavior is a very interesting problem. One of their main properties is that its generators satisfy a system of differential equations, known as Ramanujan differential equations:

$$E_2' = \frac{E_2^2 - E_4}{12}, \quad E_4' = \frac{E_2 E_4 - E_6}{3}, \quad E_6' = \frac{E_2 E_6 - E_4^2}{2},$$

where $E_k' = \frac{1}{2\pi i} \frac{dE_k}{dz} = q \frac{dE_k}{dq}$, where, again, z is a coordinate on the upper half-plane and $q = e^{2\pi i z}$.

In [55], Movasati has given an algebro-geometrical interpretation of such equations, by considering *enhanced elliptic curves*, which are triples (E, α, ω) such that α is a holomorphic 1-form (that is, α is in the first piece of the Hodge filtration), ω is a non-holomorphic $1-form$ and the intersection product $\alpha \cdot \omega = 1$. In other words, it is a curve plus a choice of basis of the middle cohomology with fixed intersection product that respects the Hodge filtration. It is easy to see that the space T of such triples is 3-dimensional and quasi-affine. One can, then, compute the Gauss-Manin connection in such basis α, ω. It can be shown that there is a unique vector field R on T such that the contraction of the Gauss-Manin connection with respect to R, i.e. ∇_R, satisfies a natural linear equation. By looking at the one-dimensional locus L of T for which R generates the tangent space and by restricting the coordinates of T to L (integral curve), we conclude that R correspond to the Ramanujan equations and the coordinates correspond to the Eisenstein series. It is important to stress that the characteristic field is computed by looking at relations among the periods of E, i.e., integrals of α and ω over integral cycles.

This gives us a natural way of generalizing quasimodular forms. The same procedure was carried out for the mirror quintic family in [56]. In the same fashion, one can consider the space of pairs (X,B), where X is a quintic in the family and B is a basis of its third de Rham cohomology respecting the Hodge structure and with constant intersection product. Then, we can find a natural vector field R similarly, by looking at periods. Those satisfy a Picard-Fuchs equation (which was first computed in the famous paper [14]) and this gives us a way of computing the Gauss-Manin connection and finding R. From that, the coordinates of the space T of such pairs (which is seven-dimensional) restricted to the locus L are functions with integral q-expansions satisfying a differential equation in the same fashion as quasimodular forms. The Yukawa coupling, which is the generating function of the Gromov-Witten invariants for the quintic, can be written in terms of these generators. In this sense, we can see a kind of modularity in the generating function of GW invariants. Chapter 1 concerns the definition of such invariants and how they can be computed for the quintic threefold.

Our contribution to the program was to consider not the simpler closed Gromov-Witten invariants, but open invariants, which are defined by looking at curves with boundaries on a Lagrangian. This corresponds to looking at periods of integrals over paths with boundaries on two conic curves inside the quintics on the mirror side. These invariants were first computed by Walcher and his collaborators [66, 62, 54]. The main difference in this case is that, instead of having a pure Hodge structure in the cohomology, we have a Mixed Hodge structure in this case. There are also technicalities in the process of defining the Gauss-Manin connection for the relative cohomology bundle. Those technicalities are treated in Chapters 2 and 3. We were able to define what it meant for a basis to be compatible with a mixed Hodge structure and defined a space T of dimension 9 of pairs (X,B), where B is such a basis with a fixed intersection product. Then, by using a nonhomogenous version of the Picard-Fuchs equation, which was first computed by Walcher in [66], we were able to compute the Gauss-Manin connection, the vector field R, and to write the generating function of open Gromov-Witten invariants in terms of the coordinate functions of T, which restricted to the locus L are complex functions satisfying a differential equation constructed in the same way as the Ramanujan equation for quasimodular forms. Details of this construction are in Chapter 5. We also give details for the analogous cases for elliptic curves and closed Gromov-Witten invariants following Movasati's papers in Chapter 4.

Refinements of Donaldson-Thomas invariants

Donaldson-Thomas (DT) invariants are invariants defined via obstruction theory, usually for Calabi-Yau threefold. Over the field of complex numbers, one can define such invariants in the following way: take Z to be the moduli spaces of subschemes N (ideal sheaves I_N) of a Calabi-Yau threefold M such that the N represents a fixed homology class β and the Euler characteristic of the sheaf $\mathscr{O}_N$ is n. Then, obstruction theory gives us a virtual class $[Z]_{vir}$ in the top cohomology, whose Poincaré dual can be integrated against the class 1 in cohomology:

$$DT(n,\beta) = \int_{[Z]^{PD}_{vir}} 1.$$

If Z ends up being smooth, the invariants are simply the topological Euler characteristics up to sign, see [50] for details. Kontsevich and Soibelman [40] introduced a refinement of such invariants that is not valued in $\mathbb{Z}$ but in (an extension of) the Grothendieck ring of varieties $K_0(Var(k))$. This ring is generated (as an abelian group) by isomorphism classes of varieties over k with the relations $[S] = [T] + [S \setminus T]$ for $T \subset S$ closed. The product is given by the Cartesian product. There is a natural morphism from this ring to $\mathbb{Z}$ given by the Euler characteristic that recovers classical DT invariants. The construction of such motivic invariants, which is related to nearby fibres, is studied in Chapter 6.

In this framework, we noticed that, by considering examples of varieties defined over $k = \mathbb{R}$ and computing the Euler characteristic with compact support of the real points, one gets real versions of DT invariants. Our main examples are the degree-zero DT invariants of $\mathbb{A}^3$. Classically, these are given by a generating series of plane partitions, and its motivic version was computed in [11]. The compactly supported Euler characteristic of the real points yields the real invariants computed in [41], which correspond to symmetric plane partitions.

Our main contribution is then to consider not only topological Euler characteristics but also to look at a quadratically refined version of it. Based on the work of Barge and Morel [8] on $\mathbb{A}^1$-homotopy theory, which, roughly, is a way of considering homotopy theory for schemes, by exchanging the interval for $\mathbb{A}^1$, one can do intersection theory with a degree function valued not in $\mathbb{Z}$, but in $GW(k)$, the ring of all quadratic forms over k. By computing the rank and signature of such forms, one recovers the classical numbers over $\mathbb{C}$ and over $\mathbb{R}$. For example, it is possible to show that, over a field k, the "number" of lines on a generic cubic threefold is a form given by $x_1^2 + \dots x_{15}^2 - x_{16}^2 - \dots x_{27}^2$ (see [36]). Of course, this has rank 27 (count over $\mathbb{C}$) and signature 15, which implies that the difference between the negative and positive terms is 3, the real signed count. This allows us to define an Euler characteristic, by looking at the degree of the Euler class on a scheme S. Another more categorical way is to consider the motivic stable homotopic category and use the fact that the infinite suspension spectrum of S is strongly dualizable and gives rise to an endomorphism of the sphere spectrum, which corresponds to an element of $GW(k)$. In Chapter 7, we give more precise definitions of $\mathbb{A}^1$-enumerative geometry, including Chow-Witt groups.

To introduce the new invariants mentioned above, we considered motivic invariants for the cases where the moduli spaces of coherent sheaves in question are given by the critical locus of a smooth function $f : X \to \mathbb{A}^1$ defined on a smooth X. In this case, these invariants are defined via the work of Denef and Loser [20] and are related to a motivic version of the Milnor fibre of f, which itself is defined as a sum over classes in $K_0(Var(k))$ of components of a strong resolution of X_0. We then introduce *arithmetic* DT invariants by taking the $\mathbb{A}^1-$version of the Euler characteristic. We also discuss the relationship between the motivic version of the Milnor fibre defined in $K_0(Var(k))$ mentioned above and the arithmetic Milnor number defined in $GW(k)$ using $\mathbb{A}^1$-enumerative geometry by Kass and Wickelgren in [35] which could give us a way to define arithmetic invariants independently. We pose the question of whether such a relationship can be found in general and, if the answer is positive, this would be a big step in explaining the relationship between the real and motivic invariants. We consider two important examples in this direction: the computation of arithmetic degree-zero DT invariants of $\mathbb{A}^3$, inspired by the work of [11] and a refined computation of Gopakumar-Vafa invariants $n_{g,d}$ for g and d at the Castelnuovo bound, using a result of [64]. All these computations, the definition of our arithmetic invariant, and the comparison with the real case are described in Chapter 8.1.

Part I

Modularity of Gromov-Witten Invariants

Chapter 1

Gromov-Witten Invariants

In this section, we give a short introduction to Gromov-Witten theory, which is the main object of study in the first part of this Thesis. Our goal here is not to prove all the results, but to give a general overview of such objects. In the following, we assume that X is a projective variety. The most interesting cases are when X is the projective space $\mathbb{P}^n$ or X is a Calabi-Yau threefold. An important aspect is that X is Kähler, and therefore it has both a symplectic and a complex/algebraic structure. On the symplectic side, Gromov-Witten theory is concerned with counting so-called J holomorphic curves, that is, maps from a Riemann surface to X that satisfy the Cauchy-Riemmann equations with respect to the complex structures. The main tools come from analysis and the study of differential equations. On the algebraic side, Gromov-Witten theory counts algebraic curves of a given genus on X and the tools come from intersection and moduli theory. Both of these are in the so-called A-side of type II string theory. Here, we focus on the algebraic definition and then we finish this chapter by mentioning Mirror Symmetry, which relates the enumerative aspects of GW theory with Hodge theoretic aspects of Calabi-Yau varieties. Mirror Symmetry was essential to first compute such invariants. We follow the book [17].

1.1 Stable Maps and Their Moduli

Given a smooth projective smooth curve C of fixed genus, and $p_1,\dots,p_n$ points on C, our goal, roughly, is to count maps $f : C \to X$, with fixed homology class $f_*[C] = \beta \in H_2(X,\mathbb{Z})$ and such that $f(p_i)$ is inside subvarieties $Z_1,\dots,Z_n$. In symbols:

$$f(p_i) \in Z_i \text{ and } f_*[C] = \beta. \tag{1.1}$$

There is a natural moduli space of pointed curves, which can be compactified (leading to the so-called "stable maps", which were first introduced in [39] and we will define shortly). The maps satisfying (1.1) form, roughly, a subvariety of the space of curves and these induce classes in the cohomology of the moduli space (denoted $\bar{M}_{g,n}$). These classes only depend on $\alpha_1,\dots,\alpha_n$, the cohomology classes associated with the Z_i and are denoted by $I_{g,n,\beta}(\alpha_1,\dots,\alpha_n)$. By integrating such classes, we get the so-called Gromov-Witten invariants.

$$GW_{g,n,\beta}(\alpha_1,\dots,\alpha_n) = \int_{\bar{M}_{g,n}} I_{g,n,\beta}(\alpha_1,\dots,\alpha_n)$$

We now devote ourselves to making such an idea more rigorous. The idea of considering morphisms from smooth curves C to X has some problems. The main one is that the moduli space of such objects is not projective and that makes intersection theory way more complicated. To get a compactified version of such spaces, we need to allow nodal curves with some conditions. Before looking at maps, we have to consider stable curves.

Definition 1.1. *An n-pointed stable curve consists of a connected marked curve* $(C, p_1, \ldots, p_n)$ *satisfying the following:*

1. *C has at most nodal singularities (double points);*
2. $p_1, \ldots, p_n$ *are all distinct smooth points;*
3. *If* C_i *is an irreducible component of C such that* $C_i \cong \mathbb{P}^1$*, then* C_i *meets the other components at more than two points.*

The last condition is equivalent to saying that the curve has only a finite number of automorphisms. Now, in the same spirit, we can define:

Definition 1.2. *An n-pointed stable map consists of a connected marked curve* $(C, p_1, \ldots, p_n)$ *and a morphism* $f : C \to X$ *satisfying the following:*

1. *C has at most nodal singularities (double points);*
2. $p_1, \ldots, p_n$ *are all distinct smooth points;*
3. *If* C_i *is an irreducible component of C such that* $C_i \cong \mathbb{P}^1$ *and* f *is constant on* C_i*, then* C_i *contains at least 3 marked or singular points;*
4. *If C has arithmetic genus 1 and* $n = 0$*,* f *is not constant.*

Remark 1.3. *The definitions above can be made relative, by considering curves* $C \to S$ *over a scheme S with sections* $s_1, \ldots s_n$ *corresponding to the marked points. Then, we ask the fibers* C_s *to have a fixed genus g for all geometric points* $s \in S$ *and, for the case of stable curves, that* C_s *is stable. For the case of stable maps, that* $f_s : C \to X$ *is stable for every geometric point* $s \in S$*. Also, if* $\beta \in H_2(X, \mathbb{Z})$*,* f *has class* β *if every* f_s *has class* β*.*

By considering the functor that takes S to the set of isomorphism classes of stable curves, we can use the classical work of [59] and [19] to define a moduli space of $\mathscr{M}_{g,n}$ of curves of genus g with n marked points. This is going to be a stack instead of a scheme, but it has an associated coarse moduli space $\bar{M}_{g,n}$. The dimension of this space is $3g - 3 + n$ and there are many ways of constructing this moduli space.

By a similar idea, the functor taking S to the set of stable maps over S of genus g and class β (the notion of isomorphism between two stable maps f_1 and f_2 are just isomorphisms g between the curves that $f_1 \circ g = f_2$), is associated to an algebraic stack $\mathscr{M}_{g,n}(X, \beta)$ representing the functor and a projective coarse moduli space $\bar{M}_{g,n}(X, \beta)$. For $g = 0$ and $X = \mathbb{P}^r$, i.e., if one is interested in counting rational curves, $\mathscr{M}_{0,n}(\mathbb{P}^r, \beta)$ is a smooth stack. This means that the associated coarse moduli space $\bar{M}_{0,n}(\mathbb{P}^r, \beta)$ is an orbifold. As in this case, the homology classes are just multiples of the class of a line ℓ, we can write $\bar{M}_{0,n}(\mathbb{P}^r, d)$, for $\beta = d\ell$. In this case, it can be shown that the dimension of this space is $rd + r + d + n - 3$. In general, the "expected dimension" of $\bar{M}_{g,n}(X, \beta)$ is given by

$$d = (1-g)(\dim X - 3) - \int_\beta \omega_X + n, \tag{1.2}$$

where ω_X is the restriction of the hyperplane section of the ambient projective space where X sits on. In other words, ω_X is the symplectic form of X.

However, $\bar{M}_{g,n}$ usually has components exceeding this dimension. This is why we need a virtual fundamental class that only takes the right components. The expected dimension is computed by considering, after fixing a curve C, the deformation of a map governed by the difference $h^1(C, f^* T_X) - h^0(C, f^* T_X)$ (normal and tangent directions). By Riemann-Roch, this is the degree of f^*T_X, which is given by the integral and $(1-g)\cdot rk(T_X) = (1-g)\dim X)$. A more rigorous way of doing this computation can be found in the work of Li and Tian [48].

With the moduli spaces $\bar{M}_{g,n}$ and $\bar{M}_{g,n}(X,\beta)$ in hand, we now turn to the goal of defining cohomology classes associated to the subvarieties appearing in the definition and virtual fundamental classes mentioned. Consider the following morphisms

$$\begin{aligned} \pi_1 : \bar{M}_{g,n}(X,\beta) &\to X^n \\ (f,C,p_1,\ldots,p_n) &\mapsto (f(p_1),\ldots,f(p_n)) \end{aligned} \tag{1.3}$$

$$\begin{aligned} \pi_2 : \bar{M}_{g,n}(X,\beta) &\to \bar{M}_{g,n} \\ (f,C,p_1,\ldots,p_n) &\mapsto (\tilde{C},p_1,\ldots,p_n), \end{aligned} \tag{1.4}$$

where $\tilde{C}$ denotes the so-called "stabilization" of the curve. This is only defined if $2g+n \geq 3$, and it is necessary because C might not be a stable curve.

If we assume that X is smooth and that $\bar{M}_{g,n}(X,\beta)$ is a smooth orbifold of expected dimension, we get natural maps in the cohomology groups:

$$\pi_1^* : H^*(X,\mathbb{Q})^{\otimes n} \to H^*(\bar{M}_{g,n}(X,\beta),\mathbb{Q})$$

$$\pi_{2,*} : H_*(\bar{M}_{g,n}(X,\beta),\mathbb{Q}) \to H_*(\bar{M}_{g,n},\mathbb{Q})$$

The second map induces, via Poincaré duality, a morphism in the cohomology

$$\pi_{2,!} : H^*(\bar{M}_{g,n}(X,\beta),\mathbb{Q}) \to H^{*+2m}(\bar{M}_{g,n},\mathbb{Q})$$

where m is the difference of dimensions of the two spaces. As the dimension of the moduli space of pointed curves is $3g-3+n$, we get $m = (g-1)\dim X + \int_X \omega_X$.

We are now in good shape to define the Gromov-Witten classes simply by composing:

$$I_{g,n,\beta}(\alpha_1,\ldots,\alpha_n) = \pi_{2,!}\pi_1^*(\alpha_1 \otimes \ldots \alpha_n) \in H^{*+2m}(\bar{M}_{g,n},\mathbb{Q}).$$

$I_{g,n,\beta}$ is a class of degree $2m + \sum \deg \alpha_i$ and, if this number is equal to $2\dim \bar{M}_{g,n} = 2(3g-3+n)$, we can compute an integral:

$$GW_{g,n,\beta}(\alpha_1,\ldots,\alpha_n) = \int_{\bar{M}_{g,n}} I_{g,n,\beta}(\alpha_1,\ldots,\alpha_n)$$

If the dimension of the moduli space is not the expected dimension, though, such integral cannot be computed and we need to integrate over a virtual fundamental class instead.

1.2 Virtual Fundamental Class

We now give a quick explanation of the definition of a virtual fundamental class. This is also important in the definition of the Donaldson-Thomas invariants from the second part of the thesis. These classes are defined when the classical fundamental classes (which are elements in the top cohomology of projective (compact) varieties) are not suitable. These classes are related to the "normal cone" associated to a subvariety. We explain the simplest case here, but refer the reader to [9] for the more general construction.

Definition 1.4. *Let Y be a smooth variety of dimension n and let $Z \subset Y$ be a subscheme and let I be its ideal sheaf. The normal cone of Z in Y is given by:*

$$C_Z Y = Spec\left(\bigoplus_{k=0}^{\infty} I^k/I^{k+1}\right)$$

When Z and Y are smooth, $C_Z Y$ is simply the total space of the normal bundle of Z inside Y. The normal cone, then, is responsible for the deformations of Z inside Y. If Z is given by zeroes of some section of a vector bundle E over Y, the normal cone can be seen inside $E|_Z$ by simply considering the map that takes a section of E and multiplying it by the section defining Z. Recall that the total space of $E|_Z$ is simply Spec($Sym(\mathscr{E}|_Z)$), where $\mathscr{E}$ is the sheaf of sections of E. Therefore, the map $\mathscr{E} \to I$ described above gives us a map $C_Z Y \to E|_Z$.

In this case, this class refines the Euler class in the following way:

Proposition 1.5 (cf. Lemma 7.1.5 of [17]). *If $Z = Z(s)$ is given by zeroes of a section s of a vector bundle $E \to Y$, we have that*

$$i_*(s^*[C_Z Y]) = c_r(E) \cap [Y]$$

where $i : Z \to Y$ is the inclusion map and $c_r(E)$ is the top Chern class of E. Here the map s^* is the Gysin map (see [27], Definition 3.3). On Chow groups, it is defined as $s^* : CH_*(E) \to CH_{*-r}(Z)$, where r is the rank, given as the inverse of the pullback of the projection, which is an isomorphism since E is a vector bundle (geometrically, this map simply contract the fibers). As $\dim C_Z Y = n$, $s^*[C_Z Y]$ is a class in $A_{n-r}Z$, which is the "expected dimension" of Z, that is, the number we would get if the section s defines Z as a complete intersection (generic case). Notice that we have abused notation and used the same notation for s and $s|_Z : Z \to E|_Z$.

This result tells us that if s is a generic section, $s^*[C_Z Y]$ is exactly $[Z]$ (by definition of the top Chern class). However, in the case in which s is not generic, we still get $s^*[C_Z Y]$ behaving like $[\tilde{Z}]$ for a generic $\tilde{Z}$, even when $i_*[Z]$ does not correspond to the Chern class (sometimes not even the dimension is right!).

Definition 1.6. *Let Z be a subscheme given as zeroes of a generic section of smooth vector bundle E over a smooth variety Y. We define $\xi = s_*([C_Z Y]) \in CH_0(Z)$ to be the virtual fundamental class of Z.*

Example 1.7. *Consider a quintic threefold $X \subset \mathbb{P}^4$. In this case, the moduli space of stable rational maps of degree zero without marked points to $\mathbb{P}^4$ (i.e. lines) is simply the Grassmaninan: $Y := \bar{M}_{0,0}(\mathbb{P}^4, \ell) = G(2,5)$. Also, the space $Z := \bar{M}_{0,0,}(X, \ell)$ can be seen as a subspace of $G(2,5)$ given as zeros of the vector bundle $E = Sym^5(Q^*)$ over $G(2,5)$, where Q is the tautological vector bundle. We can see that the inclusion $C_Z Y \to E|_Z$ gives us a zero-dimensional class $s^*[C_Z Y]$ in Z. If s is a generic section, then Z is actually dimension zero, as expected, and the degree of this class is the actual number of lines on X which is 2875. However, in cases in which s is not generic, as the Fermat quintic, X has an infinite number of lines and Z has higher dimensional components. However $s^*[C_Z Y]$ is still zero dimensional and its degree is still 2875. We do not reproduce the computation for the Fermat quintic here, but we refer the reader to [2].*

In the example above, the space $\mathscr{M}_{0,0}(X, \beta)$ could be seen as a subspace of a smooth variety. However, this is not always possible. To define virtual classes for all $M_{g,n}(X, \beta)$, one needs to use *perfect obstruction theory* to compute the *intrinsic normal cone*. This allows us to define an analog of the normal cone without considering an ambient space. This is done by Fantechi and Behrend in [9]. We won't need the details for this construction in our text. We only need to understand that the virtual fundamental class behaves like a fundamental class in the case the space has the expected dimension. Although the virtual fundamental classes constructed are in Chow groups, we consider their corresponding homology classes in $H^*(X, \mathbb{Q})$.

Now that we have the virtual fundamental class $\xi = \left[\bar{M}_{g,n}(X, \beta)\right]^{\text{virt}} \in H_d(\bar{M}_{g,n}(X, \beta))$, where d is the expected dimension as in 1.2, we can come back to the definition of Gromov-Witten invariants. Let $\pi : \bar{M}_{g,n}(X, \beta) \to X^n \times \bar{M}_{g,n}$ be the map induced by (1.3) and (1.4). Let p_1 and p_2 be the projections onto X^n and $\mathscr{M}_{g,n}$.

Definition 1.8. *Let $\beta \in H_2(X,\mathbb{Z})$ be a homology class and $\alpha_1,\ldots,\alpha_n \in H^*(X,\mathbb{Q})$ be cohomology classes, corresponding to subvarieties of X. Let $2g+n \geq 3$. Then we can define the Gromov-Witten class $I_{g,n,\beta}(\alpha_1,\ldots,\alpha_n) \in H^*\left(\bar{M}_{g,n},\mathbb{Q}\right)$ as:*

$$I_{g,n,\beta}(\alpha_1,\ldots,\alpha_n) = PD^{-1} p_{2*}\left(p_1^*(\alpha_1 \otimes \cdots \otimes \alpha_n) \cap \pi_*(\xi)\right),$$

where $\xi = \left[\bar{M}_{g,n}(X,\beta)\right]^{virt}$ is the virtual fundamental class of $\bar{M}_{g,n}(X,\beta)$ and PD is Poincaré duality.

If $n,g \geq 0$, we can define the Gromov-Witten invariant $GW_{g,n,\beta}(\alpha_1,\ldots,\alpha_n)$ as the rational number defined by

$$GW_{g,n,\beta}(\alpha_1,\ldots,\alpha_n) = \int_\xi e_1^*(\alpha_1) \cup \ldots \cup e_n^*(\alpha_n),$$

where the e_i are the components of the map π_1 in (1.3).

It is important to emphasize that although Gromov-Witten invariants are related to the count of curves on a variety, they do not always give the number of curves. Indeed, they are usually rational numbers instead of integers.

1.3 Mirror Symmetry and Computations for the Quintic Threefold

Let X be a Calabi-Yau threefold. In this situation, the physical theory predicts a duality between the so called A-model of type two string theory and B-model. The A-model is concerned mostly with enumerative and symplectic geometry and it fixes the complex structure of the variety and varies the Kähler structure. On the other hand, the B-model fixes a Kähler structure but varies the complex structure. Every Calabi-Yau manifold (that is, with fixed complex structure) has a mirror, which is a family of manifolds with fixed Kähler class for which the Physical theories (SCFTs) remain the same. For that reason, mirror manifolds have symmetric Hodge diamonds, since the dimension of the complex structure moduli of the mirror and of the Kähler moduli of the variety are the same (actually, they are locally isomorphic).

This symmetry was used to compute Gromov-Witten invariants basically by means of identifying the A-model correlation function and the B-model correlation function of the mirror. The first one is related to the variation of the Kähler moduli, while the second is related to the variation of the complex structure moduli. This allows to relate Gromov-Witten invariants to periods of varieties and variations of Hodge structures. Below, we do this for the most important example in this first part, which is the quintic threefold. For details about correlation functions the reader can check [17, Appendix B] and the references therein.

Let X denote a quintic threefold in $\mathbb{P}^4$. Recall that in this case the expected dimension of the moduli space $\bar{M}_{0,n,\beta}$ is n. Therefore, for $n=0$, the invariants can be computed even without choosing any classes α_i, since we can simply integrate 1 against the fundamental class. Denote $N_d = GW_{0,0,d\ell}$.

To compute the GW invariants of X, we first relate the actual invariants and the virtual numbers of rational curves (instanton numbers) of degree d in X, denoted by n_d. They are given by the formula

$$N_d = \sum_{k\ell=d} n_\ell k^{-3} = \sum_{k|d} n_{d/k} k^{-3}.$$

The idea behind the above formula is that, for each degree ℓ curve C on X, the component of $\bar{M}_{0,0}(X,d)$ containing the family of degree k covering maps of C contributes with a factor k^{-3}

to the value of the invariant N_d [17, Theorem 7.4.4]. Putting all the invariants together as a series:

$$\sum_{d=1}^{\infty} N_d q^d = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} n_\ell k^{-3} q^{k\ell}.$$

In order to relate the computation on the A side with the computation on the B side, we need the mirror map. This is defined in terms of periods of the mirror family corresponding to a quintic threefold in $\mathbb{P}^4$. More details on this can be seen in Section 4.2. To summarize, the mirror quintic family is a one parameter family and periods (integrals of a holomorphic 3-form with respect to a 3-cycle) are solutions of the Picard-Fuchs equation

$$\theta^4(f) - z\left(\theta+\frac{1}{5}\right)\left(\theta+\frac{2}{5}\right)\left(\theta+\frac{3}{5}\right)\left(\theta+\frac{4}{5}\right)(f) = 0,$$

where $\theta = z\frac{\partial}{\partial z}$ and z is the parameter of the family.

The mirror map T is given by the quotient of the first non-holomorphic solution of the Picard-Fuchs equation, denoted ψ_0, by the holomorphic one, denoted ψ_1. For the explicit formulas, see Theorem 4.9. On the same note, the coordinate of the Kähler moduli on the quintic is given by $q = \exp(T)$.

Relating the A-model and the B-model via Mirror Symmetry gives us that the Yukawa couplings of the A and B-models should coincide. The Yukawa coupling, on the A-side contains enumerative information while on the B-side it is intrinsic to the geometry of the variety:

$$Y_A := 5 + \sum_{d=1}^{\infty} d^3 n_d q^d, \text{ on the A model,}$$

$$Y_B := \int \Omega \wedge \Omega''', \text{ on the B model.}$$

By using the Picard-Fuchs equation, we can deduce that Y_B also satisfies a differential equation and then compute it in terms of z. Via the mirror map, we can relate q and z and find a formula for Y_A in terms of the variable q and, therefore, for the degree zero GW invariants N_d.

$$Y_A = 5 + 2875\frac{q}{1-q} + 609250\cdot 2^3\frac{q^2}{1-q^2} + \cdots + n_d d^3\frac{q^d}{1-q^d} + \ldots$$

In the next few chapters, as stated in the introduction, we focus on building a theory of modularity for the solutions of Picard-Fuchs equations and exposing some analogies between q-expansions like the formula above and q-expansions for modular forms. For the Gromov-Witten invariants as above, it was done by Movasati in [56]. Our contribution is considering the same program for open Gromov-Witten invariants for the quintic threefold, which are defined in the same way but, instead of counting closed algebraic curves or closed J-holomorphic maps, one counts J-holomorphic maps with boundary on a Lagragian, or in the algebraic setting, one counts real algebraic curves (with signs). In this setting there is also a Picard-Fuchs equation which is not homogeneous and, in the same fashion, one can relate the A and B models to get the invariants. This was first considered by Walcher in [66]. We use Hodge theoretical considerations in order to find a modular interpretation to the q-expansions computed there.

Chapter 2

Relative Cohomology in the Algebraic Setting

Classically, in the differential setting, given a family of manifolds $X \to S$, we can always look at the de Rham cohomology of each element of the family. This gives us a vector bundle of the base with a lattice given by the integral singular cohomology. Grothendieck [30], using hypercohomology, found a way to consider these objects in a purely algebraic setting, that is, when $X \to S$ is a morphism of algebraic varieties. We follow his ideas to make a similar definition not for a simple family $X \to S$, but for a situation in which each element of the family has a subvariety as boundary and the cohomology we are interested in, is the relative de Rham cohomology with boundary on the subvariety, that is, a family $Y \to X \to S$. In the same direction, one can define a connection on the de Rham cohomology bundle, which considers derivatives with respect to the coordinates of the base. Katz and Oda [37] defined an algebraic version of such a connection in the algebraic version of the above mentioned de Rham cohomology of Grothendieck. Here, we also follow their ideas to define an analogous connection for the relative version that we define.

2.1 The Algebraic Relative de Rham Cohomology

Let X be a quasi-projective variety and let $Y \subset X$ be a smooth closed subvariety over some fixed field k. Our first goal is to define what should be the algebraic relative de Rham cohomology. We recall that absolute algebraic de Rham cohomology is defined by:

$$\mathrm{H}^m_{\mathrm{dR}}(X/k) := \mathbb{H}^m(X, \Omega^\bullet_X)$$

where the $\mathbb{H}$ represents hypercohomology and $\Omega^\bullet$ is the complex of sheaves of algebraic differential forms on X. For smooth projective varieties, it is equivalent to the analytic and C^∞ de Rham cohomologies (see, for instance, chapters 4 and 5 of [58] or the article [30]). The main property of the relative cohomology is the existence of a long exact sequence for each pair (X,Y). To get the cohomology of Y, it is natural to consider the sheaf $j_*\Omega_Y$, where $j : Y \hookrightarrow X$ is the inclusion. Recall $\mathbb{H}^m(X, j_*\Omega^\bullet_Y) = \mathbb{H}^m(Y, \Omega^\bullet_Y)$. As j is a closed immersion, $j_*\Omega^\bullet_Y$ is a coherent $\mathcal{O}_X$-module.

Definition 2.1. *Let X be a quasi-projective variety and let $Y \subset X$ be a closed subvariety over some fixed field k. We define the relative de Rham complex of the pair (X,Y) as:*

$$\Omega^\bullet_{X,Y} = \mathrm{Cone}(j^\# : \Omega^\bullet_X \to j_*\Omega^\bullet_Y)[-1] = \Omega^\bullet_X \oplus \Omega^{\bullet-1}_Y$$

with coboundary operator given in terms of the coboundary operators from $\Omega^\bullet_X$ and $\Omega^\bullet_Y$ by $d(\omega,\theta) = (d\omega, j^*\omega - d\theta)$. It is useful to note that, in our case, j^* is just the restriction to Y.

The relative de Rham cohomology of the pair (X,Y) is simply the hypercohomology of this new complex:

$$H^m_{dR}(X,Y/k) := \mathbb{H}^m(X,\Omega^\bullet_{X,Y})$$

Recall that for two cochain complexes $A^\bullet, B^\bullet$ and a morphism $f : A^\bullet \to B^\bullet$ we define the cone over f denoted by Cone(f) as the cochain complex given by:

$$\mathrm{Cone}(f)^\bullet = A^{\bullet+1} \oplus B^\bullet$$

$$d(x,z) = (dx, f(x) - dz), \quad x \in A^\bullet, z \in B^\bullet$$

We get a natural short exact sequence

$$0 \to B^\bullet \to \mathrm{Cone}(f) \to A^{\bullet+1} \to 0$$

which yields a long exact sequence for which the boundary morphism $A \to B$ is induced by f. Note that the first map is not simply the inclusion but the map $b \mapsto (0,-b)$.

For more details on the homological algebra definitions, we refer to [67] and section A.1 of [63]. Notice that these references use different sign conventions and we use a third one, presented in [13]. The difference is that we consider the sign exchange in the map $B \to \mathrm{Cone}(f)$ in order to get a morphism of complexes.

2.2 The Hypercohomology Sheaf of a Family

Now, we want to deal with a family $\pi : X \to S$ of smooth projective varieties over a fixed field k and a subfamily $Y \subset X$ of closed subvarieties. We want to define a bundle over S given by the relative de Rham cohomology $\mathrm{H}^m_{\mathrm{dR}}(X,Y)$ on each fibre.

Before that, we need a general result on complexes of sheaves on families.

Definition 2.2. *Let $X \to S$ be as above and let $\mathscr{F}^\bullet$ be a complex of $\mathscr{O}_X$-modules. The m-th-hypercohomology sheaf is the sheaf of $\mathscr{O}_S$-modules associated to the presheaf given by*

$$U \mapsto \mathbb{H}^m\left(\pi^{-1}(U), \mathscr{F}^\bullet|_{\pi^{-1}(U)}\right)$$

This sheaf is the m-th hyperderived functor $R^m\pi_*$ of the direct image functor π_* from the category of complexes of $\mathscr{O}_X$-modules to complexes of $\mathscr{O}_S$-modules, which consists of applying the direct image in each element of the complex. To see that these two definitions are equivalent, one just needs to consider the Čech resolution of the complexes and then apply the functor π_*. We now proceed to show that this sheaf is coherent when the $\mathscr{F}^i$ are coherent.

Theorem 2.3. *Let $X \to S$ be a family of smooth projective varieties over a field k. Let $\mathscr{F}^\bullet$ be a bounded complex of coherent algebraic sheaves over X. Let $\mathscr{H}^m$ be its hypercohomology sheaf as defined in 2.2. Then $\mathscr{H}^m$ is coherent.*

Proof. First, we observe that $\mathscr{H}^m$ is quasi-coherent. Let U be an affine open subset of S. We will show that, over U, $\mathscr{H}^m$ is given by the sheaf (on the usual notation from [32]) $\widetilde{M}$, where $M = \mathbb{H}^m(X_U, \mathscr{F}^\bullet)$.

To show this, we just need to see that $\widetilde{M}$ and $\mathscr{H}^m$ coincide on principal open subsets of U, since these form a basis for the topology. Indeed, take any principal open subset of U, say $D(a)$. After taking an affine open covering $\mathscr{V}$ of X_U, acyclic[1] for all sheaves $\mathscr{F}^i$, (this is possible since we are assuming X to be separated), we can compute the hypercohomology via this covering. Now, to compute the hypercohomology sheaf on $D(a)$, we can intersect each

[1] An open covering $\mathscr{V}$ is acyclic with respect

element $\mathscr{V}$ with $\pi^{-1}(D(a))$. These intersections are all principal open subsets of the V_i (given by $D(b)$, where b is the image of a via the induced morphism of rings). Therefore, as the sheaves $\mathscr{F}^i$ are (quasi-)coherent, the Čech cohomology groups are all localizations of the correspondent groups that appear for the covering $\mathscr{V}$. As the localization functor is exact, we conclude that the hypercohomology group for $D(a)$ is just the localization of the cohomology group of U. This finishes the first part of the proof.

For coherence, we need to show that the cohomology modules above are finitely generated.

Let $F^i = \operatorname{Im}\left(\mathbb{H}^m\left(X_U, \mathscr{F}^{\bullet\geq i}|_{X_U}\right) \to \mathbb{H}^m\left(X_U, \mathscr{F}^{\bullet}|_{X_U}\right)\right)$ be the usual filtration of the hypercohomology. We just need to show that the quotients F^i/F^{i+1} are finitely generated.

Consider the map ϕ given by:

$$\omega = \sum_{j=i}^{m} \omega^j \in \mathbb{H}^m\left(X_U, \mathscr{F}^{\bullet\geq i}|_{X_U}\right) \xmapsto{\phi} \omega^i \in H^{m-i}\left(X_U, \mathscr{F}^i|_{X_U}\right)$$

Note that ϕ is well defined, since $D\omega = 0 \implies \delta\omega^i = 0$ and if $\omega = D\eta$, we have that $\omega^i = \delta\eta^i$ (since i is the first index). We now compute the kernel of ϕ. It is given by

$$\widetilde{F}^{i+1} := \operatorname{Im}\left(\mathbb{H}^m\left(X_U, \mathscr{F}^{\bullet\geq i+1}|_{X_U}\right) \to \mathbb{H}^m\left(X_U, \mathscr{F}^{\bullet\geq i}|_{X_U}\right)\right).$$

To see this, first notice that the image above is obviously inside the kernel since its elements can be written with $\omega^i = 0$. For the other inclusion, assume $\omega^i = \delta\eta^i$. Now, consider the element $\omega - D\eta^i$, which is another representative of ω in the hypercohomology. Notice that, in this form, the i-th term will cancel and, as $\mathrm{d}\eta^i$ is in the $(i+1)$-th part, we conclude that ω is on the image $\widetilde{F}^{i+1}$. This implies we have an isomorphism

$$\mathbb{H}^m\left(X_U, \mathscr{F}^{\bullet\geq i}|_{X_U}\right) \Big/ \widetilde{F}^{i+1} \cong H^{m-i}\left(X_U, \mathscr{F}^i|_{X_U}\right) \tag{2.1}$$

To finish, we use reverse induction on i to show that each F^i is finitely generated. Firstly, we know that there is an $i >> 0$ such that $F^i = 0$, since the complex is bounded. On the other hand, assuming that F^{i+1} is finitely generated, we can conclude the same for F^i. For this, it is enough to show that the quotient F^i/F^{i+1} is finitely generated. After factoring the natural epimorphism $\mathbb{H}^m\left(X_U, \mathscr{F}^{\bullet\geq i}|_{X_U}\right) \twoheadrightarrow F^i$ by $\widetilde{F}^{i+1}$ and its image F^{i+1}, we get another epimorphism

$$\mathbb{H}^m\left(X_U, \mathscr{F}^{\bullet\geq i}|_{X_U}\right) \Big/ \widetilde{F}^{i+1} \twoheadrightarrow F^i \Big/ F^{i+1}.$$

Using the isomorphism (2.1), we get

$$H^{m-i}\left(X_U, \mathscr{F}^i|_{X_U}\right) \twoheadrightarrow F^i \Big/ F^{i+1},$$

which reduces the proof to showing that $H^{m-i}\left(X_U, \mathscr{F}^i|_{X_U}\right)$ is finitely generated. This comes from Theorem III.8.8b from [32] and the fact that $\mathscr{F}^i$ is coherent. □

The main example of a complex of sheaves as above is the complex $\Omega^\bullet_{X/S}$ of algebraic differential forms relative to the family. In our context, we want to consider the version with boundary, $\Omega^\bullet_{X,Y/S}$, and its hypercohomology sheaf.

Definition 2.4. *For a smooth family of quasi-projective varieties $X \to S$ and $Y \subset X$ smooth, we define:*

$$\Omega^\bullet_{X,Y/S} := \operatorname{Cone}(-j^\# : \Omega^\bullet_{X/S} \to j_*\Omega^\bullet_{Y/S})[-1]$$

We also set $\mathscr{H}^q_{\mathrm{dR}}(X, Y/S)$ as the q-th hypercohomology sheaf of $\Omega^\bullet_{X,Y/S}$ as above.

2.3 Connections on (Quasi)-Coherent Sheaves

Definition 2.5. *Let X be a scheme over a field k and let $\mathscr{S}$ be a (quasi)-coherent sheaf of $\mathcal{O}_X$-modules. A connection on $\mathscr{S}$ is a morphism*

$$\rho : \mathscr{S} \to \Omega^1_{X/k} \otimes_{O_X} \mathscr{S}$$

such that, for any section f of $\mathcal{O}_X$ and any section s of $\mathscr{S}$, we have:

$$\rho(fs) = df \otimes s + f\rho(s)$$

The maps above can be extended to higher order. Indeed, we can define a map

$$\Omega^i_{X/k} \otimes_{O_X} \mathscr{S} \to \Omega^{i+1}_{X/k} \otimes_{\mathcal{O}_X} \mathscr{S}$$

by

$$\rho_i(\omega \otimes s) = d\omega \otimes s + (-1)^i \omega \wedge \rho(s)$$

where s and ω are sections on an open subset and $\omega \wedge \rho(s)$ denotes the image of $\omega \otimes \rho(s)$ under the isomorphism

$$\Omega^i_{X/k} \otimes_{O_X} \left(\Omega^1_{X/k} \otimes_{O_X} \mathscr{S}\right) \to \Omega^{i+1}_{X/k} \otimes_{\mathcal{O}_X} \mathscr{S}$$

sending $\omega \otimes \eta \otimes s$ to $(\omega \wedge \eta) \otimes s$

The most important of these higher degree maps is the one obtained for $i = 1$, used to define the so-called curvature of the connection.

Definition 2.6. *Let X and $\mathscr{S}$ be as above and ρ be a connection on $\mathscr{S}$. The curvature of the connection ρ is the morphism*

$$K = \rho_1 \circ \rho : \mathscr{S} \to \Omega^2_{X/k} \otimes_{\mathcal{O}_X} \mathscr{S}$$

A connection is called integrable or flat if $K = 0$.

We finish this section with a proposition encoding the most important property of integrable connections;

Proposition 2.7. *Let X and $\mathscr{S}$ be as above and ρ be a connection on $\mathscr{S}$. Then $\rho_{i+1} \circ \rho_i(\omega \otimes s) = \omega \wedge K(s)$. In particular, if ρ is integrable, then the sequence*

$$0 \to \mathscr{S} \xrightarrow{\rho} \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathscr{S} \xrightarrow{\rho_1} \Omega^2_{X/k} \otimes_{\mathcal{O}_X} \mathscr{S} \xrightarrow{\rho_2} \cdots$$

is a complex of sheaves.

Proof. Fix an open set $U \subset X$ and take any section s of $\mathscr{S}$. Let $\rho(s) = \sum a_j \otimes t_j$ be written as a sum of simple tensors in $\Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathscr{S}$. We compute:

$$\rho_{i+1}(\rho_i(\omega \otimes s)) = \rho_{i+1}\left(d\omega \otimes s + (-1)^i \omega \wedge \rho(s)\right) =$$

$$= \rho_{i+1}(d\omega \otimes s) + (-1)^i \rho_{i+1}(\omega \wedge \rho(s)) =$$

$$= 0 \otimes s + (-1)^{i+1} d\omega \wedge \rho(s) + (-1)^i \rho_{i+1}\left(\sum \omega \wedge a_j \otimes t_j\right) =$$

$$= (-1)^{i+1} d\omega \wedge \rho(s) + (-1)^i \Big(d\omega \wedge \rho(s) +$$

$$+ (-1)^i \omega \wedge \left(\sum da_j \otimes t_j\right) + (-1)^{i+1} \omega \wedge \left(\sum a_j \wedge \rho(t_j)\right)\Big) =$$

$$= \omega \wedge \left(\sum da_j \otimes t_j\right) + (-1)^1 \omega \wedge \left(\sum a_j \wedge \rho(t_j)\right) = \omega \wedge K(s)$$

□

2.4 Gauss-Manin Connection for the Relative Cohomology

The classical version of the Gauss-Manin connection in the algebraic setting is defined in [37]. They consider $X \to S$ a smooth morphism of smooth schemes over a field k. Then, they define a filtration on the complex $\Omega_X^\bullet$ and use the associated spectral sequence to define the connection. We will follow the same steps for the case with boundary. Let X be a smooth scheme over S and Y be a smooth closed subscheme over S, that is, we have a morphism $\pi : X \to S$ and a closed immersion $j : Y \to X$.

From [37, §2] we have natural filtrations on $\Omega_{X/k}^\bullet$ and $\Omega_{Y/k}^\bullet$ given by

$$F^p(\Omega_{X/k}^\bullet) = \operatorname{im}\left(\Omega_{X/k}^{\bullet-p} \otimes_{\mathscr{O}_X} \pi^*(\Omega_{S/k}^p) \to \Omega_{X/k}^\bullet)\right) \tag{2.2}$$

and similarly for Y.

Recall that, as j is a closed immersion, j_* is an exact functor of sheaves and therefore $j_*\Omega_{Y/k}^\bullet$ can be endowed with the direct image of the filtration from $\Omega_{Y/k}^\bullet$ (this is just the fact that exact functors preserve monomorphisms). We then define a filtration on $\Omega_{X,Y/k}^\bullet$ in terms of these two:

$$F^p\Omega_{X,Y/k}^\bullet = F^p\Omega_{X/k}^\bullet \oplus j_*\left(F^p\Omega_{Y/k}^{\bullet-1}\right).$$

As a filtered complex, $\Omega_{X,Y/k}$ has an associated spectral sequence that has page 1 given by the cohomology of the graded complex, that is,

$$E_1^{p,q} = \mathscr{H}^{p+q}\left(\mathrm{Gr}^p(\Omega_{X,Y/k}^\bullet)\right), \tag{2.3}$$

where $\mathrm{Gr}^p = F^p/F^{p+1}$ and the right hand side represents the hypercohomology sheaf of the complex (see [67]).

Proposition 2.8. *Under the same hypothesis as above, we have:*

$$\mathrm{Gr}^p(\Omega_{X,Y/k}^\bullet) = \pi^*(\Omega_{S/k}^p) \otimes_{\mathscr{O}_X} \Omega_{X,Y/S}^{\bullet-p}$$

Proof. Since exactness preserves quotients, we have:

$$\mathrm{Gr}^p(\Omega_{X,Y/k}^\bullet) = \frac{F^p\Omega_{X,Y/k}^\bullet}{F^{p+1}\Omega_{X,Y/k}^\bullet} = \frac{F^p\Omega_{X/k}^\bullet \oplus j_*\left(F^p\Omega_{Y/k}^{\bullet-1}\right)}{F^p\Omega_{X/k}^\bullet \oplus j_*\left(F^p\Omega_{Y/k}^{\bullet-1}\right)} = \mathrm{Gr}^p(\Omega_{X/k}^\bullet) \oplus j_*\mathrm{Gr}^p(\Omega_{Y/k}^{\bullet-1}) \tag{2.4}$$

As $\Omega_{X/k}$, $\Omega_{S/k}$ and $\Omega_{Y/k}$ are locally free in their respective spaces and due to the definition of the sheaves $\Omega_{X/S}$ and $\Omega_{Y/S}$, we get (see [37]):

$$\mathrm{Gr}^p(\Omega_{X/k}^\bullet) = \pi^*(\Omega_{S/k}^p) \otimes_{\mathscr{O}_X} \Omega_{X/S}^{\bullet-p} \tag{2.5}$$
$$\mathrm{Gr}^p(\Omega_{Y/k}^\bullet) = (\pi \circ j)^*(\Omega_{S/k}^p) \otimes_{\mathscr{O}_Y} \Omega_{Y/S}^{\bullet-p}. \tag{2.6}$$

Being a closed immersion, j_* commutes with tensor products and $j_*j^*(-) = - \otimes_{\mathscr{O}_X} j_*\mathscr{O}_Y$. We conclude that:

$$j_*\mathrm{Gr}^p(\Omega_{Y/k}^\bullet) = j_*(j^*\pi^*(\Omega_{S/k}^p) \otimes_{\mathscr{O}_Y} \Omega_{Y/S}^\bullet) = \pi^*(\Omega_{S/k}^p) \otimes_{\mathscr{O}_X} (j_*\mathscr{O}_Y) \otimes_{j_*\mathscr{O}_Y} j_*\Omega_{Y/S}^\bullet = \pi^*(\Omega_{S/k}^p) \otimes_{\mathscr{O}_X} j_*\Omega_{Y/S}^\bullet \tag{2.7}$$

Using the fact that tensor product and direct sums have a distributive property, from (2.4), (2.5),(2.6) and (2.7) we get the result. □

Following our plan, equation (2.3) allows us to write:

$$E_1^{p,q} = \mathscr{H}^{p+q}(\mathrm{Gr}^p(\Omega^\bullet_{X,Y/k})) = \mathscr{H}^{p+q}(\pi^*(\Omega^p_{S/k}) \otimes_{\mathscr{O}_X} \Omega^{\bullet-p}_{X,Y/S}) =$$

$$= \mathscr{H}^q(\pi^*(\Omega^p_{S/k}) \otimes_{\mathscr{O}_X} \Omega^\bullet_{X,Y/S}).$$

Using that Ω^p_S is locally free, tensoring before or after taking the hypercohomology does not matter. Also, exact funtors preserve cohomology. Then:

$$E_1^{p,q} = \Omega^p_{S/k} \otimes_{\mathscr{O}_S} \mathscr{H}^q(\Omega^\bullet_{X,Y/S}) = \Omega^p_{S/k} \otimes_{\mathscr{O}_S} \mathscr{H}^q(X,Y/S). \tag{2.8}$$

Proposition 2.9. *The morphism $d_1^{0,q} : E_1^{0,q} \to E_1^{1,q}$ is an integrable connection in the sense of 2.5 and 2.6.*

Proof. By the computation above, $d_1^{0,q}$ is a map

$$\mathscr{H}^q(X,Y/S) \to \Omega^1_{S/k} \otimes_{\mathscr{O}_S} \mathscr{H}^q(X,Y/S).$$

We just need to check it satisfies the Leibniz rule. Let f be a section in $\mathscr{O}_S$ and (ω,α) be a section in $\mathscr{H}^q(X,Y/S)$. We have, since $d_1^{0,q}$ is induced by the differential from the complexes $\Omega_{X/k}$ and $\Omega_{Y/k}$, it satisfy the Leibniz rule. Therefore:

$$d_1^{0,q}(f(\omega,\alpha)) = df \otimes (\omega,\alpha) + f d_1^{0,q}(\omega,\alpha).$$

Also, we notice that by using the same formula for the wedge product of higher degree differential forms, we conclude that $0 = d_1^{1,q} \circ d_1^{0,q}$ is the curvature of the connection. □

Definition 2.10. *The connection defined in Proposition 2.9 is going to be called the relative Gauss-Manin connection.*

Proposition 2.11. *The relative version of the Gauss-Manin connection is compatible with the long exact sequence of the pair (X,Y). In other words, the diagram*

$$\begin{array}{ccccccccc} \dots & \to & \mathscr{H}^{q-1}(Y/S) & \to & \mathscr{H}^q(X,Y/S) & \to & \mathscr{H}^q(X/S) & \to & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \to & \Omega^1_{S/k} \otimes_{\mathscr{O}_S} \mathscr{H}^{q-1}(Y/S) & \to & \Omega^1_{S/k} \otimes_{\mathscr{O}_S} \mathscr{H}^q(X,Y/S) & \to & \Omega^1_{S/k} \otimes_{\mathscr{O}_S} \mathscr{H}^q(X/S) & \to & \dots, \end{array}$$

where the vertical arrows are the Gauss-Manin connections and the other arrows are induced from the long exact sequence, commutes.

Proof. This fact is a consequence of a general fact about cone spectral sequences (see [67], exercise 5.4.4). Notice that the diagram above can be rewritten as:

$$\begin{array}{ccccccccc} \dots \to & E_1^{0,q-1}(Y) & \to & E_1^{0,q}(X,Y) & \to & E_1^{0,q}(X) & \to & E_1^{0,q}(Y) & \to \dots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \dots \to & E_1^{1,q-1}(Y) & \to & E_1^{1,q}(X,Y) & \to & E_1^{1,q}(X) & \to & E_1^{1,q}(Y) & \to \dots \end{array}$$

,

where the horizontal maps are induced by the natural cone mappings $(\omega,\alpha) \mapsto \omega$ and $\alpha \mapsto (0,\alpha)$ and by the restriction from X to Y. The vertical mappings are the Gauss-Manin connections that are induced by the spectral sequences for X and Y. Now, as these maps respect the filtrations and commute with the differential (that is, they are morphisms of filtered complexes) they induce a morphism of spectral sequences on the first level. One can check commutativity directly. □

To end this section, we state a corollary of the previous proposition that is going to be very important in our applications and is also of historical interest. Gauss-Manin connection, as we mentioned at the beginning of the chapter, should encode the way the de Rham cohomology varies in a family. For $k = \mathbb{C}$, this can be seen in terms of integrals. If we fix a homology cycle in the base S, the most natural functions to consider are integrals of differential forms (de Rham cohomology cocycles) over that cycle. Computing the derivative of such functions with respect to the coordinates in S is the same thing as computing the Gauss-Manin connection at the cocycle. For relative cycles and cocycles, we have an analogous result.

Theorem 2.12 (Integration)**.** *Let $k = \mathbb{C}$. Let $\delta_s \in H_n(X_s, Y_s)$ be a family of relative homology cycles. Consider the integration map $\int : H^n(X_s, Y_s) \times H_n(X_s, Y_s) \to \mathbb{C}$ given by*

$$(\omega, \theta) \times \delta \mapsto \int_\delta \omega - \int_{\partial\delta} \theta,$$

where the integrals on the right hand side are defined as usual, looking at a covering of the cycle and using that the algebraic de Rham cohomology is isomorphic to the usual one.

For a fixed family of cycles, and a section $\sigma \in \Gamma(\mathcal{H}^n(X, Y/S), S)$, consider the function $s \mapsto \int_{\delta_s} \sigma$. Let σ be represented by a pair (ω, θ) We can relate its differential with the Gauss-Manin connection:

$$d\left(\int_{\delta_s} (\omega, \theta)\right) = \int_{\delta_s} \nabla(\omega, -\theta)$$

where ∇ is the Gauss-Manin connection and the integration on the right hand side is made on the first term of the tensor product $\mathcal{H}^n(X, Y/S) \otimes \Omega^1_S$.

Proof. Let (ω, θ) be a cycle in the relative de Rham cohomology. We can prove the equality locally. Choose a point $0 \in S$ and a trivializing neighborhood U or the families X and Y. U which can be taken to be a product of intervals. Consider a cycle D given by the union of all δ_s for s in U. in $H_n(X)$ given by the union of all δ_s. Now, we can write, using Stokes' theorem and Fubini's theorem:

$$\int_D d\omega = \int_U \int_{\delta_s} d\omega = \int_{\partial D} \omega = \int_{\partial U} \int_{\delta_s} \omega + \int_U \int_{\partial\delta_s} \omega$$

Now, using the fact that $\partial\delta_s$ is in Y_s, we can add and subtract $d\theta$ in order to get the Gauss-Manin connection:

$$\int_U \int_{\delta_s} d\omega = \int_{\partial U} \int_{\delta_s} \omega + \int_U \int_{\partial\delta_s} (\omega|_Y + d\theta) - \int_U \int_{\partial\delta_s} d\theta$$

Using Stokes' theorem again for $d\theta$, we obtain

$$\int_U \int_{\delta_s} d\omega = \int_{\partial U} \int_{\delta_s} \omega + \int_U \int_{\partial\delta_s} (\omega|_Y + d\theta) - \int_{\partial U} \int_{\partial\delta_s} \theta$$

$$\int_{\partial U} \int_{\delta_s} \omega - \int_{\partial U} \int_{\partial\delta_s} \theta = \int_U \int_{\delta_s} d\omega - \int_U \int_{\partial\delta_s} (\omega|_Y + d\theta)$$

By Stokes on the left-hand side and the fact that the Gauss-Manin connection is simply the differential before dividing the part on Ω^1_S, we get:

$$\int_U d\left(\int_{\delta_s} \omega - \int_{\partial\delta_s} \theta\right) = \int_U d\left(\int_{\delta_s} (\omega, \theta)\right) = \int_U \int_{\delta_s} \nabla(\omega, -\theta)$$

As U is arbitrary, we conclude that the integrands must be equal. □

2.5 A (Very Detailed) Example

To end this chapter, we would like to explore a very simple case to show how all the abstract definitions we made here indeed lead us to what we expected in first place. Let $\mathbb{G}_m$ be the punctured affine line $\mathbb{A}^1 - \{0\}$. Let $X = \mathbb{G}_m \times (\mathbb{G}_m - \{1\})$ and let Y be the subscheme given by points of the form $(1,t)$ and (t,t) for $t \neq 1$. For $S = \mathbb{G}_m - \{1\}$, we consider the projection on the second coordinate $X \to S$ and the composition $Y \to S$. We are simply considering a family of punctured affine lines with two points: one being the point 1 and the other one varying with the family.

We are going to compute the relative algebraic de Rham cohomology in this case and the Gauss-Manin connection. Let us consider first the de Rham complexes of each space. Notice they are all affine schemes given by $X := \mathrm{Spec} A$, $Y := \mathrm{Spec} B$ and $S := \mathrm{Spec} C$, where

$$A := k[z,t][z^{-1},t^{-1},(t-1)^{-1}] \qquad B := \frac{A}{(z-1)(z-t)} \qquad C := k[t][t^{-1},(t-1)^{-1}].$$

As coherent sheaves over affine spaces are the same as modules over the structural ring, we can write $\Omega_X = \Omega_{A/k}$ and $\Omega_{X/S} = \Omega_{A/C}$ and the same for Y. Therefore:

$$\begin{aligned} \Omega^0_X &= A & \qquad \Omega^0_{X/S} &= A \\ \Omega^1_X &= Adz + Adt & \Omega^1_{X/S} &= Adz \\ \Omega^2_X &= Adz \wedge dt & \Omega^2_{X/S} &= 0 \end{aligned}$$

$$\begin{aligned} \Omega^0_Y &= B & \qquad \Omega^0_{Y/S} &= B \\ \Omega^1_Y &= \frac{Bdz + Bdt}{(2z-t-1)dz + (1-z)dt} = Bdt & \Omega^1_{Y/S} &= 0 \\ \Omega^2_Y &= \frac{Bdz \wedge dt}{\langle (z-t)dz \wedge dt, (1-z)dz \wedge dt \rangle} & \Omega^2_{Y/S} &= 0 \end{aligned}$$

Notice that although Y is one-dimensional, it has a non-trivial Ω^2. That is because the closure of Y in $\mathbb{A}^2$ is singular. Moreover, notice that $(2z-t-1)^2 = (t+1)^2 - 4t = (t-1)^2$ in B: this means that $2z-t-1$ is invertible. This gives us a way to write dz in terms of dt, which implies $\Omega^1_Y = Bdt$ and $\Omega^1_{Y/S} = 0$. We now turn to compute the cohomologies $H^1(X, Y/S)$ and the Gauss-Manin connection as defined above. First, we consider the filtrations in the complexes $\Omega^\bullet_X$ and $\Omega^\bullet_Y$ as defined in (2.2). We get:

$$\begin{aligned} F^0\Omega^0_X &= A & \qquad F^0\Omega^1_X &= \Omega^1_X & \qquad F^0\Omega^2_X &= \Omega^2_X \\ F^1\Omega^0_X &= 0 & F^1\Omega^1_X &= Adt & F^1\Omega^2_X &= \Omega^2_X \\ F^2\Omega^0_X &= 0 & F^2\Omega^1_X &= 0 & F^2\Omega^2_X &= 0 \end{aligned}$$

$$\begin{aligned} F^0\Omega^0_Y &= B & \qquad F^0\Omega^1_Y &= \Omega^1_Y & \qquad F^0\Omega^2_Y &= \Omega^2_Y \\ F^1\Omega^0_Y &= 0 & F^1\Omega^1_Y &= Bdt & F^1\Omega^2_Y &= \Omega^2_Y \\ F^2\Omega^0_Y &= 0 & F^2\Omega^1_Y &= 0 & F^2\Omega^2_Y &= 0 \end{aligned}$$

Notice that the bottom line is always zero, because $\Omega^2_S = 0$, and that $F^1\Omega^2$ is the whole space, because it is generated by $dz \wedge dt$ and dt comes from $\pi^*(\Omega^1_S)$.

We proceed to compute the graded pieces which are the elements $E_0^{p,q}$ of the spectral sequence associated to the filtered complex $\Omega_{X,Y}$ (whose filtration is given by the sum of the filtrations of Ω_X and Ω_Y). $E_0^{p,q} = Gr^p\Omega^{p+q}$. Looking at the filtrations from above, it is easy to see that we get the same thing as in Proposition2.8. Indeed, the differential on $\Omega_{X,Y}$ induces a map $E_0^{p,q} \to E_0^{p,q+1}$. We are especially interested in the cases $p = 0$ and $p = 1$. For $p = 0$, we have

$$0 \to E_0^{0,0} \to E_0^{0,1} \to E_0^{0,2} \to 0,$$

which translates to

$$0 \to \Omega_X^0 \to \Omega_{X/S}^1 \oplus \Omega_Y^0 \to 0 \iff 0 \to A \to Adz \oplus B \to 0$$

after using that $Gr^0\Omega_X^1 = F^0\Omega_X^1/F^1\Omega_X^1 = \Omega_X^1/Adt = Adz = \Omega_{X/S}^1$. Since $\Omega_X^0 = \Omega_{X/S}^0$ and $\Omega_Y^0 = \Omega_{Y/S}^0$ by definition, we conclude that the cohomology of the above sequence is simply the relative cohomology with respect to the family S as defined in Definition 2.1. Notice that here we do not need to compute hypercohomology, since the Čech cohomology is trivial over affine varieties.

For $p = 1$, the situation is similar. We get

$$0 \to E_0^{1,0} \to E_0^{1,1} \to 0,$$

which is simply

$$0 \to \Omega_S^1 \otimes \Omega_X^0 \to \Omega_S^1 \otimes (\Omega_X^1 \oplus \Omega_Y^0) \to 0 \iff 0 \to Adt \to (Adz \oplus B)dt,$$

after using similar arguments. As all dt parts can be sent to Ω_S^1, we end up again with $Adz \oplus B$ which corresponds to the module $\Omega_{X,Y/S}^1$.

The cohomology of the first complex is $E_1^{0,q}$ and the cohomology of the second one is $E_1^{1,q}$. It is easy to see that these are exactly the spaces expected in (2.8). For $q = 1$:

$$E_1^{0,1} = H^1(X,Y/S) = \frac{\Omega_{X/S}^1 \oplus \Omega_{Y/S}^0}{\mathrm{im}\Omega_X^0} \tag{2.9}$$

$$E_1^{1,1} = H^1(X,Y/S) \otimes \Omega_S^1 \tag{2.10}$$

Proposition 2.13. *$H^1(X,Y/S)$ is a rank two C-module, generated by the elements $\left(\frac{1}{z}dz, 0\right)$ and $(0,z)$.*

Proof. This comes from the fact that B is generated over C by 1 and z and that exact elements in $H^1(X,Y/S)$ are of the form $\left(\frac{\partial f}{\partial z}dz, -f\right)$ for $f \in A$. This gives us that, if f does not depend on z, $(0, f)$ is exact, which means that $(0, g)$ is always equivalent to $h \cdot (0,z)$ modulo exact terms, where $h \in C$. If f has a primitive F, then

$$(f dz, 0) = (0, -F) = h \cdot (0,z),$$

modulo exact forms. As the only function in A which does not have a primitive is $\frac{1}{z}$, the proof is finished. □

The Gauss-Manin connection is induced by the differential d of the original complex $\Omega_{X,Y}^\bullet$. We describe the action of the connection on the generators described above with a generic coefficient $h \in C$:

$$\nabla\left(h \cdot \frac{1}{z}dz, 0\right) = \left(-\frac{\partial h}{\partial t} \cdot \frac{1}{z}dz, h \cdot \frac{1}{z} \cdot \frac{z-1}{2z-t-1}\right) \otimes dt \in H^1(X,Y/S) \otimes \Omega_S^1$$

and

$$\nabla(0,h\cdot z)=\left(0,-\frac{\partial h}{\partial t}z-h\cdot\frac{z-1}{2z-t-1}\right)\otimes dt\in H^1(X,Y/S)\otimes\Omega^1_S,$$

where the term $\frac{1-z}{2z-t-1}$ appeared to relate dz and dt on B.

To finish the chapter, we compute periods for $k=\mathbb{C}$, that is, integrals of the cohomology with respect to homology cycles. We will check that our relative version of the Gauss-Manin connection behaves well with integrals, as expected in 2.12. Consider a path $\delta_t\in H_1(X_t,Y_t)$ connecting 1 and t for each t. We define functions on S given by pairing δ_t with the elements $\left(h\cdot\frac{1}{z}dz,0\right)$ and $(0,h\cdot z)$. Recall that, for the relative cohomology, the pairing is the natural one induced by the long exact sequence, that is,

$$\int_{[\delta]}(\omega,\theta)=\int_\delta\omega-\int_{\partial\delta}\theta,$$

where $[\delta]$ is a relative homology class.

In our case, we have:

$$\int_{[\delta_t]}\left(h\cdot\frac{1}{z}dz,0\right)=\int_1^t h\cdot\frac{1}{z}dz-0\Big|_1^t=h(t)\log(t) \tag{2.11}$$

$$\begin{aligned}\int_{[\delta_t]}\nabla\left(h\cdot\frac{1}{z}dz,0\right)&=\left(\int_{[\delta_t]}\left(-\frac{\partial h}{\partial t}\cdot\frac{1}{z}dz,h\cdot\frac{1}{z}\cdot\frac{z-1}{2z-t-1}\right)\right)dt=\\&=\left(\int_1^t-\frac{\partial h}{\partial t}\cdot\frac{1}{z}dz-\left(h\cdot\frac{1}{z}\cdot\frac{z-1}{2z-t-1}\right)\Big|_1^t\right)dt=-\frac{\partial h}{\partial t}\log(t)dt-h(t)\frac{1}{t}dt\end{aligned} \tag{2.12}$$

$$\int_{[\delta_t]}(0,hz)=\int_1^t 0-hz\Big|_1^t=h(t)-th(t) \tag{2.13}$$

$$\begin{aligned}\int_{[\delta_t]}\nabla(0,-hz)&=\left(\int_{[\delta_t]}\left(0,\frac{\partial h}{\partial t}z+h\cdot\frac{z-1}{2z-t-1}\right)\right)dt=\\&=\left(\int_1^t 0-\left(\frac{\partial h}{\partial t}z+h\cdot\frac{z-1}{2z-t-1}\right)\Big|_1^t\right)dt=\frac{\partial h}{\partial t}(t)dt-t\frac{\partial h}{\partial t}(t)dt-h(t)dt\end{aligned} \tag{2.14}$$

Notice that, computing the differential of equation (2.11), we get exactly the result in equation (2.12). This illustrates how the relative Gauss-Manin connection should behave with respect to integrals and how periods should be computed in this situation.

Chapter 3

Mixed Hodge Structure for the Relative Cohomology

3.1 Hodge Structures

In this section, we recall the basic concepts of Hodge structures and give some examples. We follow Chapter 2 of [63]. This will be the basis for defining Mixed Hodge structures on relative cohomology.

We start by recalling the basic definition of pure Hodge structure.

Definition 3.1. *Let $V_{\mathbb{Z}}$ be a $\mathbb{Z}$-module of finite rank. A* **Hodge structure of weight** *k on V is a decomposition*

$$V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} V^{p,q} \quad \text{with}\, V^{p,q} = \overline{V^{q,p}}.$$

In general, it is useful to rephrase this definition via filtrations:

Definition 3.2. *Let $V_{\mathbb{Z}}$ be a $\mathbb{Z}$-module of finite rank. A* **Hodge structure of weight** *k on V is a filtration*

$$V_{\mathbb{C}} = F^0(V) \supset \cdots \supset F^p(V) \supset F^{p+1}(V) \supset \cdots \supset F^{k+1}(V) = 0,$$

satisfying $F^p(V) \oplus \overline{F^q(V)} = V_{\mathbb{C}}$ if $p+q=k+1$.

Proposition 3.3. *Definitions 3.1 and 3.2 are equivalent.*

Proof. Let $V_{\mathbb{Z}}$ be a module as above. Consider a Hodge structure in the sense of 3.1. Define

$$F^p(V) = \bigoplus_{r \geq p} V^{r,k-r}.$$

Obviously, this definition gives us a filtration on V. To see this filtration gives us a Hodge structure in the sense of 3.2, we just need to compute $F^p(V) \cap \overline{F^q(V)}$ for $p+q=k+1$. Indeed, we get

$$F^p(V) \cap \overline{F^q(V)} = \bigoplus_{r \geq p} V^{r,k-r} \cap \bigoplus_{s \geq q} \overline{V^{s,k-s}} = \bigoplus_{\substack{r \geq p \\ s \geq q}} V^{r,k-r} \cap V^{k-s,s} = \bigoplus_{\substack{r \geq p \\ k-s \leq p-1}} V^{r,k-r} \cap V^{k-s,s}.$$

As the indices above are never equal, we conclude that the intersection is always 0. This means $F^p(V) \cap \overline{F^q(V)} = 0$ for $p+q=k+1$. Also, by swapping $\cap$ with $\oplus$ in the above argument, we get:

$$F^p(V) \oplus \overline{F^q(V)} = \bigoplus_{\substack{r \geq p \\ k-s \leq p-1}} V^{r,k-r} \oplus V^{k-s,s} = \bigoplus_{r} V^{r,k-r} = V_{\mathbb{C}}$$

On the other hand, if we have a Hodge structure in the sense of Definition 3.2, we can define

$$V^{p,q} = F^p \cap \overline{F^q}.$$

Notice that $V^{p,q} = \overline{V^{q,p}}$. To show we get a decomposition as in Definition 3.1, consider indices p, q, r and s with $r+s = p+q = k$. Without loss of generality, we can assume $p > r$ (and thus $s > r$). We get

$$V^{p,q} \cap V^{r,s} = \left(F^p \cap F^r\right) \cap \left(\overline{F^q \cap F^s}\right) = F^p \cap \overline{F^s},$$

but $p+s > r+s = k$. This means that $F^p \cap \overline{F^s} = 0$.

This shows that the sum of the $V^{p,q}$ is a direct sum. To see that this direct sum is the whole space, we need to proceed by induction. Take $x \in V_{\mathbb{C}}$ and let p be the index for which $x \in F^p(V)$ but $x \notin F^{p+1}(V)$. We proceed by decreasing induction on p. For the base case, suppose $F^{p+1}(V) = 0$. Now, we can write $x = a+b$, with $a \in F^{p+1}(V)$ and $b \in \overline{F^{k-p}(V)}$. Of course, b is also in $F^p(V)$, therefore $b \in V^{p,k-p}$. As $a = 0$, we get $x = b$. So $x \in V^{p,k-p}$.

Now, assume it is true for p+1. Again, we write $x = a+b$ as above. As $a \in F^{p+1}(V)$, by the induction hypothesis, it can be written as a sum of elements from some $V^{r,s}$. Now, as $b \in V^{p,k-p}$, we are done. This shows that $V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$. □

The main example of Hodge structure is the de Rham cohomology of a compact Kähler manifold. In this case, we have the so-called Hodge decomposition:

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C})$$

where $H^{p,q}(X,\mathbb{C})$ is the set of k-forms that are represented by forms of type (p,q). Of course, this decomposition induces a Hodge structure (see Definition 3.1).

If X is a projective smooth algebraic variety, we can define its algebraic de Rham cohomology as we did before. It is ismorphic to the usual cohomology.

$$H^m_{\mathrm{dR}}(X/\mathbb{C}) = \mathbb{H}^m(X, \Omega^\bullet_X),$$

There is a natural Hodge structure given by the filtration

$$F^i H^m_{\mathrm{dR}}(X/\mathbb{C}) = \mathrm{im}\left(\mathbb{H}^m(X, \Omega^{\bullet \geq i}_X) \to \mathbb{H}^m\left(X, \Omega^\bullet_X\right)\right),$$

where each part of the filtration is given by the image of the map induced by the natural inclusion $\Omega^{\bullet\geq i}_X \hookrightarrow \Omega^\bullet_X$. Here $\Omega^{\bullet\geq i}_X$ is zero if the index is less than i, and is isomorphic to $\Omega^\bullet_X$ if the index is greater or equal to i.

This gives us two Hodge structures on the cohomology of X, which end up being isomorphic.

Definition 3.4. *A* **morphism of Hodge structures** *is a morphism of modules $f : V \to W$ such that its complexification maps $V^{p,q}$ to $W^{p,q}$ or, in terms of filtrations, such that, its complexification maps $F^p(V)$ to $F^p(W)$.*

Remark 3.5. *If X is a smooth projective variety, both cohomologies above are isomorphic. This isomorphism is an isomorphism of Hodge structures in the sense of the definition above. For details, see [58, Chapter 5].*

It is natural to define Hodge structures in tensor products and on Hom spaces as follows:

Definition 3.6. *Let V and W be $\mathbb{Z}$-modules. Assume that they have Hodge structures of weight k and ℓ, respectively. We define:*

$$F^p(V \otimes W)_{\mathbb{C}} = \sum_m F^m(V_{\mathbb{C}}) \otimes F^{p-m}(W_{\mathbb{C}}) \subset V_{\mathbb{C}} \otimes W_{\mathbb{C}},$$

$$F^p\mathrm{Hom}(V,W)_{\mathbb{C}} := \{f : V_{\mathbb{C}} \to W_{\mathbb{C}} \mid f\left(F^n(V_{\mathbb{C}})\right) \subset F^{n+p}(W_{\mathbb{C}}), \forall n\}$$

With these definitions, $V \otimes W$ *gets a natural Hodge structure of weight* $k+\ell$ *and the space* $\mathrm{Hom}(V,W)$ *gets a Hodge structure of weight* $\ell - k$*. In particular, we can define a Hodge structure on the dual of* V*.*

Using the fact that the image of a morphism of Hodge structures has a natural Hodge structure (by taking the images of the $V^{p,q}$) and that the kernel of a morphism of Hodge structures also has (by intersecting $V^{p,q}$ with the kernel), we get the following result..

Theorem 3.7. *The category of* $\mathbb{Z}$*-modules with Hodge structures is abelian.*

3.2 Mixed Hodge Structures

In this section, our goal is to define what are mixed Hodge structures and prove basic properties. Here, we follow the Chapter 3 of [63].

Definition 3.8. *Let* V *be a finite rank* $\mathbb{Z}$*-module. A* **mixed Hodge structure** *on* V *consists of two filtrations: an increasing filtration* $W_\bullet$ *on* $V \otimes \mathbb{Q}$ *called weight filtration and a decreasing filtration* $F^\bullet$ *on* $V_{\mathbb{C}}$ *called Hodge filtration, such that* $F^\bullet$ *induces a (pure)* $\mathbb{Q}$*-Hodge structure of weight* k *on the graded pieces*

$$\mathrm{Gr}_k^W(V \otimes \mathbb{Q}) := W_k/W_{k-1}.$$

We try to give some more details about the above definition. To have a mixed Hodge structure, we not only need a decreasing filtration on the complexification (Hodge filtration), but also an increasing filtration on $V \otimes \mathbb{Q}$. The subsequent quotients of this second filtration are expected to have Hodge structures induced by the Hodge filtration. We now write what this structure should be.

In order to simplify notation, fix k and let $G = \mathrm{Gr}_k^W = W_k/W_{k+1}$. Consider $G_{\mathbb{C}}$ its complexification. It is given by the quotient $\frac{W_k \otimes \mathbb{C}}{W_{k-1}\otimes\mathbb{C}}$. It is easy to see how the Hodge filtration induces a filtration on $G_{\mathbb{C}}$:

$$F^p(G) = \frac{F^p(V) \cap (W_k \otimes \mathbb{C})}{F^p(V) \cap (W_{k-1} \otimes \mathbb{C})}$$

This expression is useful not only to see what is the structure of G but also to recover the filtration F of V once we have a Hodge structure on G. Any V with a pure Hodge structure of weight k also has a mixed Hodge structure. The weight filtration is simply given by the trivial filtration $0 = W_{k-1} \subset W_k = V$.

The cohomology ring $H^*(X,\mathbb{Z}) = \bigoplus_{j=0}^{2m} H^j(X,\mathbb{Z})$ of a compact Kähler m-manifold admits a mixed Hodge structure. We take

$$W_k = \bigoplus_{j=0}^{k} H^j(X,\mathbb{Q})$$

and

$$F^p = \bigoplus_{r \geq p} H^{r,s}(X)$$

Now, notice that $W_k/W_{k-1} = H^k(X,\mathbb{Q})$ and that

$$\frac{F^p \cap (W_k \otimes \mathbb{C})}{F^p \cap (W_{k-1} \otimes \mathbb{C})} = \bigoplus_{r \geq p} H^{r,k-r}(X)$$

which is the usual Hodge filtration.

As in the case of pure Hodge Structures, we can define morphisms.

Definition 3.9. *A* **morphism of mixed Hodge structures** *is a linear map $f : V \to W$ compatible with both filtrations W and F. Of course, it induces a morphism of Hodge structures on each graded piece Gr_k^W.*

As in the case of pure Hodge structures, it is possible to define structures on tensor products and Hom spaces. Using the same definitions given in Definition 3.6, we induce a weight and a Hodge filtration on $V \otimes W$ and $\mathrm{Hom}(V,W)$.

We now study how morphisms of mixed Hodge structures behave. In particular, we want to show that, as in the pure case, we also get an abelian category. It will be also useful for us to understand how morphisms of mixed Hodge structures behave in exact sequences.

The first problem we need to solve is the fact that, for mixed Hodge structures, we have no Hodge decomposition in the sense of Definition 3.1. This is useful to induce a Hodge structure on the image and the kernel of a mapping.

Based on the example of the cohomology ring $H^*(X,\mathbb{C})$, in which we had a decomposition $H^*(X,\mathbb{C}) = \bigoplus_{p,q\in\mathbb{Z}} H^{p,q}(X)$, we can prove the following theorem.

Theorem 3.10. *Let V be a module with a mixed Hodge structure. Then, the subspaces (after tensoring by $\mathbb{C}$)*

$$I^{p,q} := F^p \cap W_{p+q} \cap \left(\overline{F^q} \cap W_{p+q} + \sum_{j\geq 2} \overline{F^{q-j+1}} \cap W_{p+q-j} \right)$$

form a bigrading $V_{\mathbb{C}} = \bigoplus I^{p,q}$ and satisfy

$$W_k \otimes \mathbb{C} = \bigoplus_{p+q\leq k} I^{p,q} \tag{3.1}$$

$$F^p = \bigoplus_{r\geq p} I^{r,s} \tag{3.2}$$

Besides that, the $I^{p,q}$ are compatible with the subquotients $V^{p,q} \subset \mathrm{Gr}_{p+q}^W$ in the sense that the restriction of the natural projections are isomorphisms.

This decomposition is known as **Deligne splitting**.

Proof. To simplify notation, we write W_k for $W_k \otimes \mathbb{C}$ and V for $V \otimes \mathbb{C}$.

First, note that it is enough to show that the natural projections are isomorphisms in the last part of the statement above. To see this, let $x \in V$ and take k such that $x \in W_k \setminus W_{k-1}$. We can write the Hodge decomposition of W_k/W_{k-1} and pull it back to the $I^{p,q}$ (since we are assuming we have an isomorphism). Then, we use induction to show formulas (3.1) and (3.2).

We now proceed to prove that the restriction of the projection $I^{p,q} \to V^{p,q}$ is an isomorphism.

Injectivity. Notice that $I^{p,q} = F^p \cap \overline{F^q} \cap W_{p+q} \otimes \mathbb{C} \mathrm{mod} W_{p+q-2} \otimes \mathbb{C}$. Take $x \in I^{p,q}$. It can be written as a sum $x = y + w$, where $y \in F^p \cap W_{p+q} \cap \overline{F^q}$ and $w \in W_{p+q-2}$. Suppose the projection of x is zero in Gr_{p+q}^W. This means that $x \in W_{p+q-1}$ and, therefore, $y \in W_{p+q-1}$. But, as the pure Hodge structure on Gr_{p+q}^W has weight $p+q$. we conclude that $F^p \cap \overline{F^q} = 0$ in the quotient. This implies that y is actually in W_{p+q-2}. We can then assume $x = w \in W_{p+q-2}$. Repeating the argument, we can show $x = 0$ (since the filtration is zero for small indices).

Surjectivity. Take a class $[x] \in V^{p,q} \subset W_{p+q}/W_{p+q-1}$. By definition of the spaces $V^{p,q}$, there are $v \in F^p \cap W_{p+q}$ and $u \in F^q \cap W_{p+q}$ such that $[x] = [v] = [\overline{u}]$. Therefore, we can write $v = \overline{u} + w$, for some $w \in W_{p+q-1}$. Now, as Gr_{p+q-1}^W has a Hodge structure of weight $p+q-1$, we have that $F^p \cap \mathrm{Gr}_{p+q-1}^W \oplus \overline{F^q} \cap \mathrm{Gr}_{p+q-1}^W = \mathrm{Gr}_{p+q-1}^W$. This implies that $[w] = [v'] + [\overline{u'}]$, that is, $w = v' + \overline{u'} + w_1$, where $v' \in F^p \cap W_{p+q-1}$, $u' \in F^q \cap W_{p+q-1}$ and $w_1 \in W_{p+q-2}$. We can, therefore, write

$$v_1 := v - v' = \overline{u} + \overline{u'} + w_1$$
$$u_1 := u + u'$$

We end up with $v_1 = \overline{u_1} + w_1$ where $[v_1] = [\overline{u_1}] = [v] = [\overline{u}] = [x]$ in W_{p+q}/W_{p+q-1}, but $w_1 \in W_{p+q-2}$. Using the same argument, we get v_2 and u_2, but this time we need to get $u'' \in F^{q-1}$ in order to write $[w_1] = [v''] + [u'']$ in Gr^W_{p+q-2}. So we get $v_2 \in F^p \cap W_{p+q}$ and $u_2 \in F^q \cap W_{p+q} + F^{q-1} \cap W_{p+q-2}$.

Repeating as many times as necessary to get $w_N = 0$, we can define an element $y := v_N = \overline{u_N}$. Note that $[y] = [x]$ and $y \in I^{p,q}$. Indeed, we have

$$v_N \in F^p \cap W_{p+q}$$

and

$$\overline{u_N} \in \left(\overline{F^p} \cap W_{p+q} + F^{q-1} \cap W_{p+q-2} + \dots\right) = \left(\overline{F^q} \cap W_{p+q} + \sum_{j \geq 2} \overline{F^{q-j+1}} \cap W_{p+q-j}\right)$$

This means y is in the intersection of these sets, which is exactly $I^{p,q}$. Therefore, we found an element that is mapped to $[x]$ under the natural projection, which shows surjectivity. □

Lemma 3.11. *Let $f : A \to B$ be a morphism of mixed Hodge structures. Then, we have:*

$$f(A) \cap F^p(B) = f(F^p(A))$$
$$f(A) \cap W_k(B) = f(W_k(A))$$

In the general context of vector spaces with filtrations, a map satisfying the conditions above is called strict. So, we can rephrase Lemma3.11 by saying that any morphism of mixed Hodge structures is strict with respect to both filtrations W and F.

Proof. First, note that we always have an inclusion $f(F^p(A)) \subset f(A) \cap F^p(B)$, since f, being a morphism of Hodge structures, satisfy $f(F^p(A)) \subset F^p(B)$. The same is of course true for W.

For the other inclusion, we use Theorem 3.10. Observe that f satisfy $f(I_A^{p,q}) \subset I_B^{p,q}$, since it preserves the filtrations. Take $x \in f(A) \cap F^p(B)$ and $y \in A$ such that $f(y) = x$. Suppose $y \notin F^p(A)$. Then, we can take $y \in F^m(A)$, with $m < p$. Write $y = \sum_{r \geq m} y^{r,s}$ via Deligne splitting. Now, we have a decomposition $x = \sum_{r \geq m} f(y^{r,s})$, with $f y^{r,s} \in I_B^{r,s}$. As $x \in F^p(B)$, we have that $f(y^{r,s}) = 0$ for $m \leq r < p$. Now, take $y' = \sum_{r \geq p} y^{r,s}$. Clearly, $f(y') = x$ and $y' \in F^p(A)$. Therefore $x \in f(F^p(A))$. A similar argument shows the fact for the filtration W. □

Corollary 3.12. *Let*

$$A \to B \to C$$

be an exact sequence of $\mathbb{Z}$-modules equipped with mixed Hodge structures. Then, for all k, p, the sequences

$$\mathrm{Gr}^W_k A_{\mathbb{Q}} \to \mathrm{Gr}^W_k B_{\mathbb{Q}} \to \mathrm{Gr}^W_k C_{\mathbb{Q}} \tag{3.3}$$
$$\mathrm{Gr}^p_F A_{\mathbb{C}} \to \mathrm{Gr}^p_F B_{\mathbb{C}} \to \mathrm{Gr}^p_F C_{\mathbb{C}} \tag{3.4}$$
$$\mathrm{Gr}^p_F \mathrm{Gr}^W_k A_{\mathbb{C}} \to \mathrm{Gr}^p_F \mathrm{Gr}^W_k B_{\mathbb{C}} \to \mathrm{Gr}^p_F \mathrm{Gr}^W_k C_{\mathbb{C}} \tag{3.5}$$

are also exact.

The corollary is a direct consequence of the previous lemma. It allows us to put a natural mixed Hodge structure on the image of a morphism of Hodge structures since it allows us to put a pure Hodge structure on the graded pieces. Also, as the exactness is preserved, it is clear that the quotient of a mixed Hodge structure by the kernel of a morphism is the image of this morphism.

We conclude the following:

Corollary 3.13. *The category of mixed Hodge structures is abelian.*

We now prove a proposition that will be our main ingredient to define a mixed Hodge structure on the relative cohomology.

Proposition 3.14. *Let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence of $\mathbb{Q}$-vector spaces with an increasing filtration $W_\bullet$ and a decrasing filtration $F^\bullet$. Suppose the maps are strict with respect to both filtrations and that they induce mixed Hodge structures on A and C. Then the filtrations induce a mixed Hodge structure on B.

Proof. It is sufficient to prove that there is a pure Hodge structure induced by the filtration F on the graded pieces $\mathrm{Gr}_k^W B = W_k(B)/W_{k-1}(B)$. By 3.12 and the fact the maps are strict w.r.t W, we have an exact sequence

$$0 \to \mathrm{Gr}_k^W A \to \mathrm{Gr}_k^W B \to \mathrm{Gr}_k^W C \to 0,$$

where the first and last terms have pure Hodge structures of weight k. Using that the maps are strict w.r.t F, we can restrict this sequence to

$$0 \to A_k^{p,q} \to B_k^{p,q} \to C_k^{p,q} \to 0,$$

where $*_k^{p,q} = F^p(\mathrm{Gr}_k^W(*)) \cap \overline{F^q(\mathrm{Gr}_k^W(*))}$. This implies that $\bigoplus B_k^{p,q} = \mathrm{Gr}_k^W B$, since $\bigoplus A_k^{p,q} = \mathrm{Gr}_k^W A$ and $\bigoplus C_k^{p,q} = \mathrm{Gr}_k^W C$, which shows that F induces a Hodge structure on $\mathrm{Gr}_k^W B$. Therefore, B has a mixed Hodge structure.

□

Remark 3.15. *It is possible to show that the converse of Proposition 3.14 is also true, that is, if B admits a mixed Hodge structure, the maps are strict with respect to both filtrations. Of course, this is a simple consequence of Theorem 3.10.*

3.3 The Case of Relative de Rham Cohomology

In this section, let X be a smooth projective variety (or even a compact Kähler manifold). Assume we have $Y \subset X$ a smooth closed subvariety (closed submanifold). We want to understand the mixed Hodge structure in the relative algebraic de Rham cohomology $H^m_{dR}(X,Y)$, as defined in Chapter 2.

The first step in order to describe the mixed Hodge structure on the relative cohomology is to give a Hodge filtration. The natural way to do it is to consider, for each p, the complex

$$(F^p\Omega^\bullet_{X,Y})^k = \begin{cases} 0, & k < p \\ \Omega^k_X, & k = p \\ \Omega^k_{X,Y}, & k > p \end{cases}$$

and take

$$F^p H^m_{dR}(X,Y) = \mathrm{im}(\mathbb{H}^m(X, F^p\Omega^\bullet_{X,Y}) \to \mathbb{H}^m(X, \Omega^\bullet_{X,Y})).$$

At first, it may seem strange to consider Ω_X as the first nonzero term. However, this is natural: we are simply considering the direct sum $F^p\Omega^\bullet_X \oplus F^p\Omega^{\bullet-1}_Y$. This is the usual way to induce a filtration in the cone of a morphism (see [63], Theorem 3.22). Next, we observe that this filtration can be related to the filtrations on $H^m_{dR}(X)$ and $H^m_{dR}(Y)$. Fix a good covering $\mathscr{U}$. An element in $H^m_{dR}(X,Y)$, by the definition of hypercohomology, is represented by a sum $\sigma = \sum_{i=0}^m \sigma^i$ with

$$\sigma^i = (\omega^i, \alpha^i),\ \omega^i \in C^{m-i}(\mathscr{U}, \Omega^i_X),\ \alpha^i \in C^{m-i}(\mathscr{U}, \Omega^{i-1}_Y),$$

which is D-closed. It is important to observe that we are applying the fact that $C^k(\mathscr{U},\Omega^j_{X,Y}) = C^k(\mathscr{U},\Omega^j_X) \oplus C^k(\mathscr{U},\Omega^{j-1}_Y)$, which is true by definition of $\Omega_{X,Y}$. If $\sigma \in F^p H^m_{dR}(X,Y)$, we have $\sigma^k = 0$ for $k < p$ and $\sigma^p = (\omega^p, 0)$. This means that the elements from $F^p H^m_{dR}(X,Y)$ can be represented by a pair (ω,α) of sums with $\omega \in F^p \mathscr{L}^m(\mathscr{U},\Omega^\bullet_X)$ and $\alpha \in F^p \mathscr{L}^m(\mathscr{U},\Omega^\bullet_Y)$, where $\mathscr{L}^\bullet$ is the total complex from the definition of hypercohomology.

Lemma 3.16. *Let $Y \subset X$ be a smooth closed subvariety of a smooth projective variety (or a closed submanifold of a compact Kähler manifold). Then the natural filtration on the relative cohomology is related to the filtrations of $H^m_{dR}(X)$ and $H^m_{dR}(Y)$ in the following way:*

$$\alpha(F^p(H^{m-1}_{dR}(Y)) = \mathrm{Im}(\alpha) \cap F^p(H^m_{dR}(X,Y))$$

$$\beta(F^p(H^m_{dR}(X,Y)) = \mathrm{Im}(\beta) \cap F^p(H^m_{dR}(X)).$$

Here α and β are the maps appearing in the long exact sequence.

The proof follows from the observation above and the fact that the maps α and β are induced by the inclusion $x \mapsto (0,-x)$ and the projection $(x,y) \mapsto x$.

We can put all these considerations together in one theorem.

Theorem 3.17. *Let $Y \subset X$ be a closed subvariety of a smooth projective variety (or a closed submanifold of a compact Kähler manifold). Then the relative cohomology $H^m_{dR}(X,Y)$ has a mixed Hodge structure with filtrations given by:*

$$W_k H^m_{dR}(X,Y) = \begin{cases} 0, & k < m-1 \\ \mathrm{im}\alpha, & k = m-1 \\ H^m_{dR}(X,Y), & k \geq m \end{cases}$$

$$F^p H^m_{dR}(X,Y) = \mathrm{im}\left(\mathbb{H}^m(X, F^p\Omega^\bullet_{X,Y}) \to \mathbb{H}^m\left(X,\Omega^\bullet_{X,Y}\right)\right)$$

This structure makes the long exact sequence of the pair an exact sequence of mixed Hodge structures.

Proof. The theorem is a direct application of Proposition 3.14. Consider the short exact sequence:

$$0 \to \mathrm{im}\alpha \xrightarrow{\iota} H^m_{dR}(X,Y) \xrightarrow{p} \ker(|_Y) \to 0 \tag{3.6}$$

Note that $\mathrm{im}\alpha$ and $\ker|_Y$ have pure Hodge structures. We denote $A = \mathrm{im}\alpha$ and $C = \ker|_Y$. A has a pure Hodge structure of weight $m-1$ and Hodge filtration given by $F^p A = \alpha(F^p H^{m-1}_{dR}(Y))$. On the other hand, C has a pure Hodge structure of weight m, with Hodge filtration given by $F^p C = F^p H^m_{dR}(X) \cap C$.

We are now in conditions to verify the hypothesis of Proposition3.14. By lemma 3.16, the morphisms ι and p are strict w.r.t the filtrations. Indeed,

$$\iota(F^p A) = \alpha(F^p H^{m-1}_{dR}(Y)) = F^p H^m_{dR}(X,Y) \cap \mathrm{im}\alpha = F^p(H^m_{dR}(X,Y)) \cap \mathrm{im}\iota$$

and

$$p(F^p H^m_{dR}(X,Y)) = \beta(F^p H^m_{dR}(X,Y)) = F^p C \cap \mathrm{im}\beta = F^p C \cap \ker|_Y$$

This shows our result.

□

Chapter 4

Gauss-Manin Connection in Disguise: Generalizing quasimodularity

Quasimodular forms are holomorphic functions on the upper half-plane which can be given by polynomials in terms of the Eisenstein series E_2, E_4, E_6. This is a natural extension of the ring of modular forms (which is generated by only E_4 and E_6) and one of its main properties is that its generators satisfy the system of differential equations

$$E_2' = \frac{E_2^2 - E_4}{12}, \quad E_4' = \frac{E_2E_4 - E_6}{3}, \quad E_6' = \frac{E_2E_6 - E_4^2}{2},$$

where $E_k' = \frac{1}{2\pi i}\frac{dE_k}{dz} = q\frac{dE_k}{dq}$.

In [55], Movasati has given an algebro-geometrical interpretation of such equations, by considering *enhanced elliptic curves*, which are triples (E, α, ω), where E is an elliptic curve, α is a holomorphic 1-form (that is, α is in the first piece of the Hodge filtration), ω is a non-holomorphic 1-form and $\alpha \cdot \omega = 1$. In other words, it's a curve plus a choice of basis of the middle cohomology with fixed intersection product that respects the Hodge filtration. The space T of such triples is 3 dimensional and quasi-affine. One can then compute the Guass-Manin connection in such basis α, ω. Then it can be shown that there is a unique vector field on T such that the Gauss-Manin connection satisfies a natural linear equation. Then, by looking at the one-dimensional locus L of T generated by the integral curves for R and by restricting the coordinates of T to L, we conclude that R is the Ramanujan equations and the coordinates correspond to the Eisenstein series. It is important to stress that the vector field is computed by looking at relations among the periods of E, i.e., integrals of α and ω over integral cycles.

In the same fashion, as it is done in [56], one can consider the space of pairs (X, B), where X is a quintic in the family and B is a basis of its third de Rham cohomology respecting the Hodge structure and with constant inner product. Then, we can find a natural vector field R similarly, by looking at periods. These periods satisfy a Picard-Fuchs equation (which was first computed in the famous paper [14]) and this gives us a way of computing the Gauss-Manin connection and finding R. From that, the coordinates of the space T of such pairs (which is seven-dimensional) restricted to the locus L are functions with integral q-expansions satisfying a differential equation in the same fashion as quasimodular forms. The Yukawa coupling, which is the generating function of the Gromov-Witten invariants for the quintic, can be written in terms of these generators. In this sense, we can see a kind of modularity in the generating function of GW invariants.

This idea of using the Gauss-Manin connection to find a geometric interpretation of quasimodular forms and later to generalize it to more general settings is called *Gauss-Manin connection in disguise*. This name comes from the fact that, in principle, the Gauss-Manin connection is "hidden" and appears only after you look at the geometric setting.

4.1 Elliptic Curves and Quasimodular Forms

In this section, for completion, we reproduce the results present in the work [55], where the geometrization of quasimodular forms was first considered.

4.1.1 Weierstrass Family of elliptic curves

Let our base field be $\mathbb{C}$.

Recall that any elliptic curve can be written in the form:

$$E : y^2 = 4(x-t_1)^3 - t_2(x-t_1) - t_3 \tag{4.1}$$

which is known as the Weierstrass form. This curve is smooth if and only if $\Delta := t_2^3 - 27t^3 \neq 0$. Recall also that $\mathrm{H}^1_{\mathrm{dR}}(E)$ is two-dimensional. We can then choose a basis of differential forms α and ω for which $<\omega,\alpha>= 1$. Here, $<,>$ is the intersection product:

$$<\omega,\alpha>= \frac{1}{2\pi i}\int_E \omega\wedge\alpha$$

If we choose α to be a regular form given by $\frac{\mathrm{d}x}{y}$ in the coordinates above, we can take ω to be $x\alpha$. It is possible to prove that:

$$<\omega,\alpha>= 1$$

With this in hands, we can define $T := \mathrm{Spec}(\mathbb{C}[t_1,t_2,t_3,\frac{1}{\Delta}])$ and consider $X \to T$ with X being the family of curves E equipped with forms as above. Notice that, although t_1 seems unnecessary, but, if we consider an equation without t_1, this variable would appear as parametrizing a family of differential forms,

$$\omega_{t_1} = x\frac{\mathrm{d}x}{y} + t_1\frac{\mathrm{d}x}{y},$$

for which we get $\langle\omega_{t_1},\alpha\rangle = 1$.

By a change of coordinates $x \mapsto x - t_1$, we get the third parameter as above.

4.1.2 The Gauss-Manin Connection

We want to study periods on E, that is, integrals of differential forms over cycles. In particular, we want to study:

$$\int_\delta \frac{\mathrm{d}x}{y} \quad\text{and}\quad \int_\delta \frac{x\mathrm{d}x}{y},$$

where $\delta \in \mathrm{H}_1(E)$. Notice that these periods give rise to regular functions on T. Thinking analytically, T is an open set in $\mathbb{C}^3$ and these functions are holomorphic maps. The Gauss-Manin connection gives relations between these integrals:

$$\begin{bmatrix} \mathrm{d}\left(\int_\delta \frac{\mathrm{d}x}{y}\right) \\ \mathrm{d}\left(\int_\delta \frac{x\mathrm{d}x}{y}\right)\end{bmatrix} = \nabla_{GM}\begin{bmatrix} \int_\delta \frac{\mathrm{d}x}{y} \\ \int_\delta \frac{x\mathrm{d}x}{y}\end{bmatrix}$$

where ∇_{GM} is the Gauss-Manin connection written as a matrix of differential forms in $\Omega^1(T)$ in the coordinates t_1,t_2,t_3. It is given by:

$$\nabla_{GM} = \frac{1}{\Delta}\begin{pmatrix} -\frac{3}{2}t_1\theta - \frac{1}{12}d\Delta & \frac{3}{2}\theta \\ \Delta dt_1 - \frac{1}{6}t_1 d\Delta - \left(\frac{3}{2}t_1^2 + \frac{1}{8}t_2\right)\theta & \frac{3}{2}t_1\theta + \frac{1}{12}d\Delta \end{pmatrix}$$

$$\Delta = 27t_3^2 - t_2^3, \theta = 3t_3dt_2 - 2t_2dt_3$$

∇_{GM} can be regarded as a connection on the so-called de Rham vector bundle on T with fibers given by the de Rham cohomology of each curve (seen as a point of T). But, for now, it suffices to see it as the relation between the derivatives with respect to coordinates t_1, t_2, t_3 and the differential forms α and ω.

4.1.3 Period Map and Ramanujan Vector Field

Choose two cycles δ_1 and δ_2 which form a symplectic basis of the homology with $\langle\delta_1, \delta_2\rangle = -1$. For each elliptic curve of the family 4.1, one can compute the periods in the matrix:

$$\begin{pmatrix} \int_{\delta_1} \frac{dx}{y} \int_{\delta_1} \frac{xdx}{y} \\ \int_{\delta_2} \frac{dx}{y} \int_{\delta_2} \frac{xdx}{y} \end{pmatrix} \tag{4.2}$$

It can be proven that this matrix belongs to the following set:

$$\mathscr{P} := \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \mid x_i \in \mathbb{C}, x_1x_4 - x_2x_3 = 1, \Im(x_1\overline{x_3}) > 0 \right\},$$

which has two natural actions. One left-SL$(2,\mathbb{Z})$ action and a right action of the group G, which is given by:

$$G := \left\{ \begin{pmatrix} r & s \\ 0 & r^{-1} \end{pmatrix} \middle| \; r \in \mathbb{C}^* \text{ and } s \in \mathbb{C} \right\}$$

Note that G also acts on T by multiplying the differential forms:

$$g \bullet (E, \alpha, \omega) = (E, r\alpha, s\alpha + r^{-1}\omega)$$

Lemma 4.1. *In the chart (t_1, t_2, t_3), the action is given by:*

$$g \bullet (t_1, t_2, t_3) = (r^{-2}t_1 + sr^{-1}, r^{-4}t_2, r^{-6}t_3).$$

Proof. Indeed, these coordinates correspond to the curve:

$$E : y^2 - 4\left(x - sr^{-1} - t_1r^{-2}\right)^3 + t_2r^{-4}\left(x - sr^{-1} - t_1r^{-2}\right) + t_3r^{-6} = 0.$$

and the forms $\alpha = \frac{dx}{y}$ and $\omega = \frac{xdx}{y}$. Multiplying the equation by r^6, we get:

$$E : (r^3y)^2 - 4\left(xr^2 - sr - t_1\right)^3 + t_2\left(r^2x - sr - t_1\right) + t_3 = 0.$$

This means that the triple (E, α, ω), after the change of coordinates $x' = r^2x - rs$ and $y' = r^3y$, is the curve (E', α, ω). It suffices to verify that $\alpha = r\alpha'$ and $\omega = s\alpha' + r^{-1}\omega'$, where $\alpha' = \frac{dx'}{y'}$ and $\omega' = \frac{x'dx'}{y'}$, since (E', α', ω') corresponds to coordinates (t_1, t_2, t_3).

$$\alpha' = \frac{\mathrm{d}(r^2x - rs)}{r^3y} = \frac{r^2\mathrm{d}x}{r^3y} = r^{-1}\frac{\mathrm{d}x}{y} = r^{-1}\alpha \implies \alpha = r\alpha'$$

$$\omega' = \frac{(r^2x - rs)\mathrm{d}(r^2x - rs)}{r^3y} \implies r^4\frac{x\mathrm{d}x}{r^3y} - sr^3\frac{\mathrm{d}x}{r^3y} = r\frac{x\mathrm{d}x}{y} - s\frac{\mathrm{d}x}{y} = r\omega - s\alpha$$

As $\alpha = r\alpha'$, we have:

$$\omega' = r\omega - sr\alpha' = r(\omega - s\alpha') \implies \omega = r^{-1}\omega' + s\alpha'$$

□

After factoring by the $\mathrm{SL}(2,\mathbb{Z})$ action, we can define a map:

$$\rho : T \to \frac{\mathscr{P}}{\mathrm{SL}(2,\mathbb{Z})}$$

$$(E,\alpha,\omega) \mapsto \frac{1}{\sqrt{-2\pi i}} \begin{pmatrix} \int_{\delta_1} \alpha & \int_{\delta_1} \omega \\ \int_{\delta_2} \alpha & \int_{\delta_2} \omega \end{pmatrix}$$

which is called **period map.**

Theorem 4.2. *The period map ρ is a local biholomorphism and satisfy:*

$$\rho(g \bullet K) = \rho(K)g, \quad K \in T,\ g \in G.$$

Proof. To see that it is a local biholomorphism, we just need to compute the derivative and show it is invertible. This can be done via the Gauss-Manin connection since we are differentiating through the integral sign.

For the other statement, we simply compute:

$$\rho(K)g = \frac{1}{\sqrt{-2\pi i}} \begin{pmatrix} \int_{\delta_1} \alpha & \int_{\delta_1} \omega \\ \int_{\delta_2} \alpha & \int_{\delta_2} \omega \end{pmatrix} \cdot \begin{pmatrix} r & s \\ 0 & r^{-1} \end{pmatrix} = \frac{1}{\sqrt{-2\pi i}} \begin{pmatrix} \int_{\delta_1} r\alpha & \int_{\delta_1} s\alpha + r^{-1}\omega \\ \int_{\delta_2} r\alpha & \int_{\delta_2} s\alpha + r^{-1}\omega \end{pmatrix} = \rho(g \bullet K)$$

□

Now, let $\mathbb{H}$ be the upper half plane $\{z \in \mathbb{C} \mid \Im(z) > 0\}$. Observe that $\mathbb{H}$ can be included in $\mathscr{P}$ by the map $z \mapsto \left(\begin{smallmatrix} z & -1 \\ 1 & 0 \end{smallmatrix}\right)$. $\mathbb{H}$ is special in this context because it can be shown that the orbits of G are in bijection with $\mathbb{H}$, that is, any element can be written in the form $\left(\begin{smallmatrix} z & -1 \\ 1 & 0 \end{smallmatrix}\right)$ after multiplication by $g \in G$. In other words, this is the same as saying that each point of $\mathbb{H}$ corresponds to an elliptic curve (since we are identifying all the enhancements). Using that ρ is a local biholomorphism and that $\mathbb{H}$ is simply connected, we can define a map

$$\gamma = (\gamma_1, \gamma_2, \gamma_3) : \mathbb{H} \to T$$

after lifting the inclusion $\mathbb{H} \to \frac{\mathscr{P}}{\mathrm{SL}(2,\mathbb{Z})}$

Lemma 4.3. *The coordinates of γ will satisfy functional equations:*

$$(cz+d)^{-2i} \gamma_i \left(\frac{az+b}{cz+d} \right) = \gamma_i(z), \quad i = 2,3$$

and

$$(cz+d)^{-2} \gamma_1 \left(\frac{az+b}{cz+d} \right) = \gamma_1(z) - c(cz+d)^{-1}, z \in \mathbb{H}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

This functional equations are exactly the equations satisfied by quasimodular forms, which show that γ_i are quasimodular forms.

Proof. To see this, one could use the previous theorem. Firstly, we write $\frac{az+b}{cz+d}$ as a matrix and realize that it can be written as a product:

$$\begin{pmatrix} \frac{az+b}{cz+d} & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} az+b & -a \\ cz+d & -c \end{pmatrix} \cdot \begin{pmatrix} (cz+d)^{-1} & c \\ 0 & cz+d \end{pmatrix} =$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (cz+d)^{-1} & c \\ 0 & cz+d \end{pmatrix}$$

This means that, by the theorem above, we can take h out of the brackets. Modulo the left action of $\mathrm{SL}(2,\mathbb{Z})$, we have that

$$\gamma\left(\frac{az+b}{cz+d}\right)=\rho^{-1}\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}z & -1\\ 1 & 0\end{pmatrix}\right)\bullet h=\gamma(z)\bullet h$$

where $h=\begin{pmatrix}(cz+d)^{-1} & c\\ 0 & cz+d\end{pmatrix}$ and we have used the previous theorem.

To end, it suffices to use the formula that we have for the action in coordinates, and then isolate $\gamma_i(z)$.

$$\gamma(z)\bullet h=\left((cz+d)^2\gamma_1(z)+c(cz+d),(cz+d)^4\gamma_2(z),(cz+d)^6\gamma_3(z)\right)$$

□

To finish this section, we want to understand what would be the pushforward of the vector field given by the derivative $\frac{\partial}{\partial z}$ in $\mathbb{H}$ by the map γ, in order to find nice differential equations.

Our first idea is to identify $\frac{\partial}{\partial z}$ on $\mathbb{H}$ with $\frac{\partial}{\partial x_1}$ on $\mathscr{P}$. But one can rapidly notice that the second vector field is not $\mathrm{SL}(2,\mathbb{Z})$ invariant and, therefore, is not defined on the quotient. To correct this, we consider the vector field:

$$X=-x_2\frac{\partial}{\partial x_1}-x_4\frac{\partial}{\partial x_3}$$

which is invariant under the action and, when we restrict ourselves to $\mathbb{H}$ corresponds to $\frac{\partial}{\partial z}$. Let R be the pullback of the vector field X by ρ (or the pushforward by γ), then:

$$R\int_{\delta_i}\alpha=-\int_{\delta_i}\omega$$

and

$$R\int_{\delta_i}\omega=0$$

In terms of the Gauss-Manin connection:

$$\begin{aligned}\nabla_R\alpha&=\omega\\ \nabla_R\omega&=0\end{aligned}$$

By direct computation, we find that there is only one vector field that satisfies the two equalities above and it is given by:

$$R=\left(t_1^2-\frac{1}{12}t_2\right)\frac{\partial}{\partial t_1}+(4t_1t_2-6t_3)\frac{\partial}{\partial t_2}+\left(6t_1t_3-\frac{1}{3}t_2^2\right)\frac{\partial}{\partial t_3}$$

The vector field R is called **Ramanujan vector field** and, as it corresponds to $\frac{\partial}{\partial z}$ via γ, we get the differential equations:

$$\begin{aligned}\frac{d\gamma_1}{dz}&=\gamma_1^2-\frac{1}{12}\gamma_2,\\ \frac{d\gamma_2}{dz}&=4\gamma_1\gamma_2-6\gamma_3,\\ \frac{d\gamma_3}{dz}&=6\gamma_1\gamma_3-\frac{1}{3}\gamma_2^2\end{aligned}$$

4.1.4 The Picard-Fuchs Equation and Eisenstein Series

To relate what we did before with the Eisenstein series, we need to solve the Ramanujan differential equations. This was first done by Ramanujan himself, who verified that the famous Eisenstein series satisfy the equations above. These are given by:

$$E_{2i}(q) := \left(1 + b_i \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2i-1}\right) q^n\right), \quad i = 1,2,3$$
$$(b_1, b_2, b_3) = (-24, 240, -504)$$

These three series form a basis for the space of solutions of the Ramanujan system of differential equations. We are now going to explain how to obtain them from the period map and to get another explicit formula for them.

To compute some of the above periods explicitly, we consider a one-parameter family of elliptic curves:

$$E_\psi : y^2 = 4x^3 + 12x - 4\psi \tag{4.3}$$

By making use of the Gauss-Manin connection, it is possible to see that the periods satisfy differential equations. For instance, $\int_{\delta_i} \frac{\mathrm{d}x}{y}$ satisfy:

$$\frac{5}{36} I + 2\psi I' + (\psi^2 - 4) I'' = 0 \tag{4.4}$$

and an analogous statement is true for ω. This equation is known as **Picard-Fuchs equation**. From all we did above, it is possible to recover the famous Eisenstein series, which generate the space of (quasi-)modular forms. First, we solve the Picard-Fuchs equation 4.4.

Theorem 4.4. *The Picard-Fuchs equation 4.4 has a basis of two solutions: one is holomorphic at $\psi = 2$ and the other one at $\psi = -2$. If we change coordinates $z = \frac{\psi-2}{4}$, we get holomorphic solutions at 0 and 1. These solutions correspond to the periods $I_2 = \int_{\delta_2} \alpha$ and $I_1 = \int_{\delta_1} \alpha$, respectively, and can be explicitly computed:*

$$\int_{\delta_2} \frac{dx}{y} = \frac{\pi}{\sqrt{3}} F\left(\frac{1}{6}, \frac{5}{6}, 1 \,|\, \frac{\psi+2}{4}\right) = \frac{\pi}{\sqrt{3}} F\left(\frac{1}{6}, \frac{5}{6}, 1 \,|\, z\right),$$
$$\int_{\delta_1} \frac{dx}{y} = \frac{i\pi}{\sqrt{3}} F\left(\frac{1}{6}, \frac{5}{6}, 1 \,|\, \frac{-\psi+2}{4}\right) = \frac{i\pi}{\sqrt{3}} F\left(\frac{1}{6}, \frac{5}{6}, 1 \,|\, 1-z\right),$$

where

$$F(a,b,c \,|\, z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

with $(a)_n = a(a+1)(a+2)\dots(a+n-1)$ and $(a)_0 = 1$.

Proof. We can use the Frobenius method, since all the coefficients are holomorphic. We compute the first solution, which is holomorphic at zero. After writing the solution as a power series

$$I(z) = \sum_{n=0}^{\infty} A_n z^n$$

and substituting into the differential equation, we get the following computation

$$\frac{5}{36} I + (2z-1) I' + z(z-1) I'' = 0 \implies$$
$$\implies \frac{5}{36} A_n + 2nA_n - (n+1)A_{n+1} + n(n-1)A_n - (n+1)nA_{n+1} = 0$$

Now, we get a recursion:

$$A_{n+1} = \frac{5 + 36n + 36n^2}{36(n+1)^2} A_n$$

To solve this recursion, we need to compute A_0, which is simply the integral for $\psi = -2$:

$$\int_2^\infty \frac{\mathrm{d}x}{\sqrt{4x^3+12-8}} = \int_2^\infty \frac{\mathrm{d}x}{\sqrt{4(x+1)^2(x-2)}} = \int_2^\infty \frac{\mathrm{d}x}{2(x-1)\sqrt{x-2}} = \frac{\pi}{\sqrt{3}}$$

With the recursion and the first term, it is easy to prove by induction that:

$$\int_{\delta_2} \frac{\mathrm{d}x}{y} = \frac{\pi}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(n!)^2} z^n$$

Now, we repeat the process for a series $\sum A_n(z-1)^n$. □

Using the same idea for ω, we get the matrix below, after solving another differential equation:

$$\begin{pmatrix} \int_{\delta_1} \frac{dx}{y} \int_{\delta_2} \frac{dx}{y} \\ \int_{\delta_1} \frac{xdx}{y} \int_{\delta_2} \frac{xdx}{y} \end{pmatrix} = \begin{pmatrix} \frac{\pi i}{\sqrt{3}} F\left(\frac{1}{6}, \frac{5}{6}, 1 \mid 1-z\right) & \frac{\pi}{\sqrt{3}} F\left(\frac{1}{6}, \frac{5}{6}, 1 \mid z\right) \\ \frac{\pi i}{\sqrt{3}} F\left(\frac{-1}{6}, \frac{7}{6}, 1 \mid 1-z\right) - \frac{\pi}{\sqrt{3}} F\left(\frac{-1}{6}, \frac{7}{6}, 1 \mid z\right) \end{pmatrix}$$

But how do the Eisenstein series come into play? We again make use of the relationship of the period map and the action of the group G as we presented in Theorem 1.2. We can write:

$$\frac{1}{\sqrt{-2\pi i}} \begin{pmatrix} \int_{\delta_1} \alpha & \int_{\delta_1} \omega \\ \int_{\delta_2} \alpha & \int_{\delta_2} \omega \end{pmatrix} \begin{pmatrix} \left(\frac{1}{\sqrt{-2\pi i}} \int_{\delta_2} \alpha\right)^{-1} & -\frac{1}{\sqrt{-2\pi i}} \int_{\delta_2} \omega \\ 0 & \frac{1}{\sqrt{-2\pi i}} \int_{\delta_2} \alpha \end{pmatrix} = \begin{pmatrix} \frac{\int_{\delta_1} \alpha}{\int_{\delta_2} \alpha} & -1 \\ 1 & 0 \end{pmatrix}$$

This implies that, for any triple $(E, \alpha, \omega) \in T$ given by coordinates (t_1, t_2, t_3), we have:

$$\gamma\left(\frac{\int_{\delta_1} \alpha}{\int_{\delta_2} \alpha}\right) = (t_1, t_2, t_3) \bullet \begin{pmatrix} \left(\frac{1}{\sqrt{-2\pi i}} \int_{\delta_2} \alpha\right)^{-1} & -\frac{1}{\sqrt{-2\pi i}} \int_{\delta_2} \omega \\ 0 & \frac{1}{\sqrt{-2\pi i}} \int_{\delta_2} \alpha \end{pmatrix} =$$

$$\left(-t_1(2\pi i)^{-1}\left(\int_{\delta_2} \alpha\right)^2 + (2\pi i)^{-1} \int_{\delta_2} \omega \int_{\delta_2} \alpha,\ t_2 \cdot (2\pi i)^{-2}\left(\int_{\delta_2} \alpha\right)^4, -t_3(2\pi i)^{-3}\left(\int_{\delta_2} \alpha\right)^6\right)$$

In the case of the family E_ψ, we have $t_1 = 0$, $t_2 = 12$ and $t_3 = -4\psi$. Therefore, we get:

$$\gamma\left(\frac{\int_{\delta_1} \alpha}{\int_{\delta_2} \alpha}\right) = \left((2\pi i)^{-1} \int_{\delta_2} \omega \int_{\delta_2} \alpha,\ 12 \cdot (2\pi i)^{-2}\left(\int_{\delta_2} \alpha\right)^4, 4\psi(2\pi i)^{-3}\left(\int_{\delta_2} \alpha\right)^6\right)$$

It is natural now to consider the new coordinate $\tau := \frac{\int_{\delta_1} \alpha}{\int_{\delta_2} \alpha}$. Actually, we can consider τ as a map $\psi \mapsto \tau(\psi) = \frac{\int_{\delta_1} \alpha}{\int_{\delta_2} \alpha}$. In this case, τ is called **mirror map** or **Schwartz map**. Combining everything, we get the following result:

Theorem 4.5. *After changing coordinates from ψ to τ, we get that:*

$$\left(\int_{\delta_2} \alpha\right)(\tau) \cdot \left(\int_{\delta_2} \omega\right)(\tau) = 2\pi i E_2(\tau)$$

$$\left(\int_{\delta_2} \alpha\right)^4 (\tau) = 4\pi^2 E_4(\tau)$$

$$(1-2\psi)\left(\int_{\delta_2} \alpha\right)^4 (\tau) = -8\pi^3 i E_6(\tau)$$

Proof. The proof is based on the fact that the Eisenstein series are solutions of the Ramanujan differential equations. Then, the coordinates of $\gamma(\tau)$ are going to be the Eisenstein series in the coordinate τ. The rest is just a question of isolating the periods. □

The lesson we take from all this is that, in general, it might be interesting to look at the quotient of solutions of other Picard-Fuchs differential equations (for example in the case of the Mirror Quintic as below) and use this as a new coordinate. Of course, we will not always get that this quotient is in the upper half plane, but we can always define $q = e^{2\pi i \tau}$ and make q expansions.

4.2 Mirror Quintic Family and Calabi-Yau Modular Forms

Here, we reproduce the results from the paper [56] and the book [57]. Those are the generalization of quasimodular forms to the mirror quintic family.

4.2.1 The Mirror Quintic Family

In this section, we consider the family of threefolds which is the mirror, in the sense of Mirror Symmetry, to a general quintic.

Definition 4.6. *Let $\psi^5 \neq 1$ and let G be the group given by*

$$G = \left\{ (a_0, \dots, a_4) \in \mathbb{Z}_5^5 : \sum_i a_i \equiv 0 \bmod 5 \right\} \Big/ \mathbb{Z}_5 , \tag{4.5}$$

where $\mathbb{Z}_5$ is embedded diagonally. This group acts on $\mathbb{P}^4$ in the natural way:

$$(a_0, \dots, a_4) \bullet [x_0, \dots, x_4] \mapsto [\mu^{a_0} x_0 : \dots \mu^{a_4} x_4],$$

where μ is a primitive fifth root of the unit. For us, a **mirror quintic** *X_ψ is the resolution of singularities of the quotient*

$$\left\{ [x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{P}^4 \mid x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0 \right\} / G. \tag{4.6}$$

After the quotient and the resolution, one can observe that the varieties obtained are Calabi-Yau. For details, see [28] or [17, Section 2.2]. For this family, one could compute the rank of its cohomologies. We get that $\dim \mathrm{H}^3_{\mathrm{dR}}(X) = 4$ and $\dim \mathrm{H}^2_{\mathrm{dR}}(X) = 101$. Its hodge diamond is given by:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} .$$

Employing the same process explained above for the elliptic curves, we consider the parameter space of enhanced mirror quintics.

Definition 4.7. *An enhanced mirror quintic consists of a pair (X,B), where X is an element of the mirror quintic family and $B=\{\alpha_1,\dots,\alpha_4\}$ is a basis of its third algebraic de Rham cohomology, such that*

$$\alpha_i \in F^{4-i}H^3_{dR}(X)\setminus F^{5-i}H^3_{dR}(X),\ \forall i>0; \quad and \quad [\langle \alpha_i,\alpha_j\rangle]=\Phi;$$

where

$$\Phi=\begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\end{bmatrix}$$

Proposition 4.8. *The parameter space for enhanced mirror quintics is given by:*

$$T=Spec\left(\mathbb{C}\left[t_0,t_1,t_2,t_3,t_4,t_5,t_6,\frac{1}{t_5t_4(t_4-t_0^5)}\right]\right).$$

Proof. Firstly, let us consider only the space of pairs (X,ω), where ω is a holomorphic 3-form on X. There are affine coordinates for this space: one can associate coordinates (t_0,t_4) to a mirror quintic X_ψ, with $\psi^{-5}=\frac{t_4}{t_0^5}$.

$$X_{t_0,t_4} := \{f(x)=0\}/G \tag{4.7}$$
$$f(x) := -t_4x_0^5-x_1^5-x_2^5-x_3^5-x_4^5+5t_0x_0x_1x_2x_3x_4. \tag{4.8}$$

In the affine space $x_0=1$, the 3-form dependent on (t_0,t_4) is the form induced on the resolution of the quotient via the residue form

$$\omega_1 := \frac{dx_1\wedge dx_2\wedge dx_3\wedge dx_4}{df}$$

which is invariant after the action of the group G. Note that these coordinates are only defined if $t_0^5\neq t_4$ and $t_4\neq 0$. Now, starting from ω, via derivation w.r.t t_0, we can get a basis W, since each derivative is an element of the next piece of the Hodge Filtration. Then, we write a base change matrix:

$$A=\begin{bmatrix} 1 & 0 & 0 & 0\\ a & b & 0 & 0\\ c & t_6 & t_5 & 0\\ t_1 & t_2 & t_3 & d\end{bmatrix}$$

By using that the intersection product must be constant and by computing the intersection product of the basis W, one can get relations among a,b,c,d and the t_i. That gives us the 6 coordinates. As t_5 has to be different from 0 (otherwise the matrix is not a change of basis), we get the result. □

4.2.2 The Gauss-Manin Connection and the Picard-Fuchs Equation

Again, as in the case of elliptic curves, we have the Gauss-Manin connection acting on the cohomology bundle and we can think of periods as functions on T. They satisfy a differential equation computed in [14] known as the Picard-Fuchs equation and the Gauss-Manin connection can be computed from it. If $I=\int_\delta\omega$, where ω is a regular 3-form and δ is a 3 cycle, we have that I satisfy

$$\theta^4-z\left(\theta+\frac{1}{5}\right)\left(\theta+\frac{2}{5}\right)\left(\theta+\frac{3}{5}\right)\left(\theta+\frac{4}{5}\right)=0$$

where $\theta = z\frac{\partial}{\partial z}$ and z is the parameter of the family and correspond to ψ^{-5}.

It is important to note that I is a function in T, that is, it is a function only on the coordinate z, since it only depends on the regular form ω and not on the other forms of the basis. To compute the Gauss-Manin conection, we start from ω and consider the basis of the cohomology given by $\frac{\partial^i \omega}{\partial z^i} = \nabla^i_{\frac{\partial\omega}{\partial z}}\omega$. Then the Gauss-Manin connection in this basis is simply the matrix whose first three rows are [0100], [0010] and [0001] and the last row is the coefficients of the equation. Then, as we want to consider the connection in the space T, we need to change to a basis satisfying the conditions in the definition 4.7 using the matrix given in (4.2.1).

Solving the equation above by the Frobenius method and making a base change, we get a basis of the space of solutions $\psi_i(z)$, $i = 1,2,3,4$ which are exactly the periods $I_i = \int_{\delta_i}\omega$, where δ_i form a symplectic basis of the homology. These solutions can be made explicit:

Theorem 4.9. *The solutions of the Picard-Fuchs differential equation are given by hypergeometric power series as follows:*

$$\psi_i(z) = \frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}\left(5^{-5\varepsilon}F(\varepsilon,z)\right),\Big|_{\varepsilon=0} \qquad j = 0,1,2,3,$$

where

$$F(\varepsilon,z) := \sum_{n=0}^{\infty}\frac{\left(\frac{1}{5}+\varepsilon\right)_n\left(\frac{2}{5}+\varepsilon\right)_n\left(\frac{3}{5}+\varepsilon\right)_n\left(\frac{4}{5}+\varepsilon\right)_n}{(1+\varepsilon)_n^4}z^{\varepsilon+n}$$

Proof. The idea is, as before, to write the solution as a series and substitute it in the equation. Then prove by induction that the recursion relation between the coefficients gives us the formulas above.

□

4.2.3 The Period Map and the Vector Field

We start by choosing a symplectic basis of the homology $H_3(X,\mathbb{Z})$, for each element of the mirror quintic family. In the same way as in (4.2), we define:

$$P = \left[\int_{\delta_i}\alpha_j\right],$$

where α_j are elements of a basis satisfying the conditions of Definition 4.7. Again using the properties of the basis and the fact that the δ_i's form a symplectic basis, we get many polynomial relations among the entries of P. This allows us to define a period domain $\mathscr{P}$. There is an action of the symplectic group $Sp(4,\mathbb{Z})$ on $\mathscr{P}$ that controls the choice of the symplectic basis. There is also an action of the algebraic group G, which is the group that acts on T by changing the basis.

$$G := \left\{g = [g_{ij}]_{4\times 4} \in \mathrm{GL}(4,\mathbb{C}) \mid g_{ij} = 0, \text{ for } j < i \text{ and } g^{\mathrm{t}}\Phi g = \Phi\right\},$$

$$\left\{g = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ 0 & g_{22} & g_{23} & g_{24} \\ 0 & 0 & g_{33} & g_{34} \\ 0 & 0 & 0 & g_{44}\end{pmatrix}, g_{ij}\in\mathbb{C}\left(\begin{matrix} g_{11}g_{44} = 1, \\ g_{22}g_{33} = 1, \\ g_{12}g_{44} + g_{22}g_{34} = 0, \\ g_{13}g_{44} + g_{23}g_{34} - g_{24}g_{33} = 0,\end{matrix}\right)\right\}.$$

One can see that there is a natural one-dimensional locus $L \subset \mathscr{P}$ which is in bijection with the orbits of G. The locus L is given by matrices of the form (τ's can be written in terms of periods):

$$\tau = \begin{pmatrix} \tau_0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \tau_1 & \tau_3 & 1 & 0 \\ \tau_2 & -\tau_0\tau_3 + \tau_1 & -\tau_0 & 1 \end{pmatrix},$$

with relations $\tau_3 = \frac{\partial \tau_1}{\partial \tau_0}, \frac{\partial \tau_2}{\partial \tau_0} = \tau_1 - \tau_0 \frac{\partial \tau_1}{\partial \tau_0}$.

Restricted to this locus, if one computes the Gauss-Manin connection restricted to the vector field $\frac{\partial}{\partial \tau_0}$, we get the matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\partial \tau_3}{\partial \tau_0} & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Pulling back to T and using the explicit formula for the Gauss-Manin connection, we get the existence of a unique vector field R and function Y for which

$$\nabla_R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & Y & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The vector field R is given by the following system of differential equations, where t_i are the coordinates of T.

$$\begin{aligned}
\frac{\partial}{\partial \tau} t_0 &= q\frac{\partial}{\partial q} t_0 = \frac{1}{t_5}\left(6\cdot 5^4 t_0^5 + t_0 t_3 - 5^4 t_4\right) \\
\frac{\partial}{\partial \tau} t_1 &= q\frac{\partial}{\partial q} t_1 = \frac{1}{t_5}\left(-5^8 t_0^6 + 5^5 t_0^4 t_1 + 5^8 t_0 t_4 + t_1 t_3\right) \\
\frac{\partial}{\partial \tau} t_2 &= q\frac{\partial}{\partial q} t_2 = \frac{1}{t_5}\left(-3\cdot 5^9 t_0^7 - 5^4 t_0^5 t_1 + 2\cdot 5^5 t_0^4 t_2 + 3\cdot 5^9 t_0^2 t_4 + 5^4 t_1 t_4 + 2 t_2 t_3\right) \\
\frac{\partial}{\partial \tau} t_3 &= q\frac{\partial}{\partial q} t_3 = \frac{1}{t_5}\left(-5^{10} t_0^8 - 5^4 t_0^5 t_2 + 3\cdot 5^5 t_0^4 t_3 + 5^{10} t_0^3 t_4 + 5^4 t_2 t_4 + 3 t_3^2\right) \\
\frac{\partial}{\partial \tau} t_4 &= q\frac{\partial}{\partial q} t_4 = \frac{1}{t_5}\left(5^6 t_0^4 t_4 + 5 t_3 t_4\right) \\
\frac{\partial}{\partial \tau} t_5 &= q\frac{\partial}{\partial q} t_5 = \frac{1}{t_5}\left(-5^4 t_0^5 t_6 + 3\cdot 5^5 t_0^4 t_5 + 2 t_3 t_5 + 5^4 t_4 t_6\right) \\
\frac{\partial}{\partial \tau} t_6 &= q\frac{\partial}{\partial q} t_6 = \frac{1}{t_5}\left(3\cdot 5^5 t_0^4 t_6 - 5^5 t_0^3 t_5 - 2 t_2 t_5 + 3 t_3 t_6\right)
\end{aligned}$$

Also, Y can be proved, via computations with periods, to be the Yukawa coupling that, in coordinates can be computed as follows:

$$\frac{5^8\left(t_4 - t_0^5\right)^2}{t_5^3}$$

Finally, restricting to L, we can solve the system of equations associated to R.

Theorem 4.10. *The seven functions*

$$\begin{aligned}
t_0 &= x_{21} \\
t_1 &= 5^4 x_{21}((6z-1)x_{21} + 5(11z-1)x_{22} + 25(6z-1)x_{23} + 125(z-1)x_{24}) \\
t_2 &= 5^4 x_{21}^2((2z-7)x_{21} + 15(z-1)x_{22} + 25(z-1)x_{23}) \\
t_3 &= 5^4 x_{21}^3((z-6)x_{21} + 5(z-1)x_{22}), \\
t_4 &= z x_{21}^5 \\
t_5 &= 5^5(z-1)x_{21}^2(x_{12}x_{21} - x_{11}x_{22}) \\
t_6 &= 5^5(z-1)x_{21}(3(x_{12}x_{21} - x_{11}x_{22}) + 5(x_{13}x_{21} - x_{11}x_{23}))
\end{aligned}$$

are holomorphic at $z = 0$ and satisfy the system of differential equations associated with R, where x_{ij} are functions of the solutions of the Picard-Fuchs equation given by:

$$\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 5 & \frac{5}{2} & -\frac{25}{12} \\ -5 & 0 & -\frac{25}{12} & 200\frac{\zeta(3)}{(2\pi i)^3} \end{pmatrix} \begin{pmatrix} \frac{1}{5^4}\psi_3 \\ \frac{2\pi i}{5^4}\psi_2 \\ \frac{(2\pi i)^2}{5^4}\psi_1 \\ \frac{(2\pi i)^3}{5^4}\psi_0 \end{pmatrix}.$$

The proof of the theorem above is purely computational. The functions on L are not quasi-modular forms as in the case of the elliptic curve, but satisfy a system of differential equations: they are, therefore, generalizations of modular forms. Following this idea, the t_i would correspond to Eisenstein series. We now move to the computations of the invariants.

To finish, the lesson we took from all this is that we can, starting from a family and basis for each de Rham cohomology, we can consider periods in this basis, find a natural one-dimensional locus in the period domain, pull the vector space on this locus back and find a unique vector space on T. This gives us a system of differential equations that explains why t_i's are generalizations of modular forms. Restricting to L, one can use the mirror map to get q-expansions for t_i. In the next chapter, we do the same thing but for the case of Open Gromov-Witten invariants.

Chapter 5

The Case of Open Gromov-Witten Invariants

In the previous chapter, we considered a geometric interpretation for quasimodular forms and explained how to generalize this idea to get Calabi-Yau modular forms. Our contribution is to execute the ideas from the GMCD program for the case of Open Gromov-Witten invariants. As explained in chapter 1, these invariants count curves with boundary on a Lagrangian in a generic quintic threefold and can be computed by looking at "relative periods" of the mirror quintic family with boundary on the union of two conic curves. This means that instead of considering absolute cohomology and the Gauss-Manin connection on the mirror quintic family, we take a relative version of the algebraic de Rham cohomology $\mathrm{H}^3_{\mathrm{dR}}(X, C_+ \cup C_-)$ and a relative version of the Gauss-Manin connection defined in Chapter 2. Then, we construct a moduli space T_{op} of triples $(W, C_\pm, [\alpha_0, \ldots, \alpha_4])$, where W is an element of the mirror family, $C_\pm$ is the pair of conic curves on each element of the family and $[\alpha_0, \ldots, \alpha_4]$ is a basis of the third relative algebraic de Rham cohomology which respects the **mixed** Hodge structure that can be defined on the relative cohomology as in Chapter 3. Before us, the idea of considering mixed Hodge structures and relative cohomology in the GMCD framework was examined in the case of elliptic curves with two fixed points in the paper [15], in which the authors recovered the theory of Jacobi forms of index zero, and in the paper [5], where the authors considered affine Calabi-Yau varieties.

5.1 Results

Definition 5.1. *A relatively enhanced mirror quintic is simply a triple*

$$(X, C_\pm, [\alpha_0, \ldots, \alpha_4]),$$

where X is a mirror quintic, $C_\pm$ is the pair of homologous curves cited above and specified in (5.4) *and $[\alpha_0, \ldots, \alpha_4]$ is a basis of $H^3_{dR}(X, C_+ \cup C_-)$ satisfying the following properties. Let δ_0 be any homology class connecting the two curves. Then the properties read:*

(i) $\alpha_i \in F^{4-i} \setminus F^{5-i}, \quad i > 0$;

(ii) $[\langle \alpha_i, \alpha_j \rangle] = \Phi$;

(iii) $\alpha_0 \in F^1 \setminus F^2$;

(iv) $\alpha_0 \in W_2$;

(v) $\int_{\delta_0} \alpha_0 = 1$;

(vi) $\alpha_i \in W_3 \setminus W_2, \quad i > 0$.

where $\Phi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}$.

Although condition (v) above seems to depend on the choice of δ_0, it is an algebraic condition that is not influenced by this choice. Indeed, α_0 ends up being a class on the image of the map $\mathrm{H}^2_{\mathrm{dR}}(C_+ \cup C_-) \to \mathrm{H}^3_{\mathrm{dR}}(W, C_+ \cup C_-)$ and, therefore, its integral over any homology class depends only on the boundary of the homology class.

Theorem 5.2. *Relatively enhanced mirror quintics can be parametrized by the nine coordinates in affine space given by*

$$\mathsf{T}_{\mathsf{op}} := Spec\left(\mathbb{C}\left[s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, \frac{1}{s_5(s_0^{10} - s_4^{10})s_0 s_4}\right]\right). \tag{5.1}$$

For an explicit description of the parametrization above, see the proof of Theorem 5.2 in Section 5.3.1. With this moduli space in hands, it is possible to compute a differential equation relating these generators s_i with open Gromov-Witten invariants and the virtual count of disks.

Theorem 5.3. *Consider the space* T_{op} *defined above. Let* A *be the Gauss-Manin connection matrix in the basis* α. *There is a unique vector field* R, *for which the Gauss-Manin connection composed with it is given, in the basis* α, *by*

$$\mathsf{A}_{\mathsf{R}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \mathsf{F} & 0 & 0 & \mathsf{Y} & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

for regular functions Y *and* F *in* T_{op}. *The expression of* R, F *and* Y *in the coordinates given in Theorem 5.2 are*

$$\mathsf{Y} = \frac{5^8\left(s_4^{10} - s_0^{10}\right)^2}{s_5^3}, \quad \mathsf{F} = -s_7\mathsf{Y}, \tag{5.2}$$

$$\mathsf{R} : \begin{cases} \dot{s}_0 = \dfrac{1}{2s_0 s_5}\left(6\cdot 5^4 s_0^{10} + s_0^2 s_3 - 5^4 s_4^{10}\right) \\ \dot{s}_1 = \dfrac{1}{s_5}\left(-5^8 s_0^{12} + 5^5 s_0^8 s_1 + 5^8 s_0^2 s_4^{10} + s_1 s_3\right) \\ \dot{s}_2 = \dfrac{1}{s_5}\left(-3\cdot 5^9 s_0^{14} - 5^4 s_0^{10} s_1 + 2\cdot 5^5 s_0^8 s_2 + 3\cdot 5^9 s_0^4 s_4^{10} + 5^4 s_1 s_4^{10} + 2s_2 s_3\right) \\ \dot{s}_3 = \dfrac{1}{s_5}\left(-5^{10} s_0^{16} - 5^4 s_0^{10} s_2 + 3\cdot 5^5 s_0^8 s_3 + 5^{10} s_0^6 s_4^{10} + 5^4 s_2 s_4^{10} + 3s_3^2\right) \\ \dot{s}_4 = \dfrac{1}{10 s_5}\left(5^6 s_0^8 s_4 + 5 s_3 s_4\right) \\ \dot{s}_5 = \dfrac{1}{s_5}\left(-5^4 s_0^{10} s_6 + 3\cdot 5^5 s_0^8 s_5 + 2 s_3 s_5 + 5^4 s_4^{10} s_6\right) \\ \dot{s}_6 = \dfrac{1}{s_5}\left(3\cdot 5^5 s_0^8 s_6 - 5^5 s_0^6 s_5 - 2 s_2 s_5 + 3 s_3 s_6\right) \\ \dot{s}_7 = -s_8 \\ \dot{s}_8 = -\dfrac{5^{12}\left(s_0^{10} - s_4^{10}\right)}{s_5}\cdot\dfrac{15}{8}\left(\dfrac{s_4}{s_0}\right)^5 \dfrac{1}{25\sqrt{5}} \end{cases} \tag{5.3}$$

Finally, looking at the above equation, we have that each s_j has a weight, given by the degree of each equation:

$$w_0 = \frac{1}{2}, w_1 = 2, w_2 = 3, w_3 = 4, w_4 = \frac{1}{2}, w_5 = 2, w_6 = 1, w_7 = -1, w_8 = 1.$$

The proof of Theorem 5.3 is computational. The interesting part about it is that, if we consider s_i functions on a variable q, take the derivation to be $5q\frac{d}{dq}$, fix initial values $s_{0,0} := \frac{1}{\sqrt{5}}$, $s_{0,1} := 12\sqrt{5}$ and $s_{0,4} := 0$ and allow fractional exponents, we get

$$-5^3\mathsf{Y} = 5 + 2875q + 4876875q^2 + 8564575000q^3 + \cdots = \sum_{d=0}^{\infty} n_d d^3 \frac{q^d}{1-q^d}$$

$$\frac{4}{5^3}\mathsf{F}(q) := 30q^{1/2} + 13800q^{3/2} + 27206280q^{5/2} + 47823842250q^{7/2} + \ldots =$$

$$= \sum_{d\,\mathrm{odd}} n_d^{\mathrm{disk}} d^2 \frac{q^{d/2}}{1-q^d},$$

where n_d are the virtual counts of rational curves of degree d on a generic quintic threefold (see [14] and [56]) and n_d^{disk} are the virtual counts of disks with boundary on a lagragian of a quintic threefold (see [66] and [62]). The q-expansions for all functions s_i can be found on the author's webpage[1]. Notice that s_1, s_2, s_3, s_4, s_5 and s_6 are the same as the corresponding t_i from [56, Theorem 3] and that $s_0^2 = t_0$ and $s_4^{10} = t_4$.

5.2 Moduli Space

From the Elliptic curve case and the computation of closed Gromov-Witten invariants in Chapter 4, our goal is to construct an analogous moduli space. We are going to assign coordinates for the space of triples $(X_\psi, C_\pm, \omega)$, where ω is a holomorphic differential 3-form on X_ψ. From Section 4.2.1, we already have coordinates t_0 and t_4 corresponding to X and ω. The curves $C_\pm$ also depend on these coordinates, but, to avoid taking tenth and square roots, we introduce new coordinates s_0 and s_4 which satisfy $s_0^2 = t_0$ and $s_4^{10} = t_4$. In these coordinates the curves are the resolution of singularities of the quotient of

$$C_\pm = \left\{ s_4^2 x_0 + x_1 = 0, x_2 + x_3 = 0, s_4 {x_4}^2 \pm \sqrt{5} s_0 x_1 x_3 = 0 \right\} \subset X_{t_0,t_4}, \tag{5.4}$$

by the group G. In the appendix of [54], they give an explicit way to solve these singularities. We can, therefore, associate a pair (s_0, s_4) to a triple $(X_{s_0,s_4}, C_\pm, \omega)$. Of course, this association is only defined when $s_4^{10} \neq s_0^{10}$ and $s_4 s_0 \neq 0$, since for $s_0 = 0$ both curves C_+ and C_- are equal. We end up with a quasi affine space $\mathsf{S}_{\mathrm{op}} := \mathbb{C}^2 \setminus \{s_0 s_4 (s_0^{10} - s_4^{10}) = 0\}$ parametrizing the triples $(X_{s_0,s_4}, C_\pm, \omega)$. This parametrization has an important property which we state below:

Proposition 5.4. *Let $r \in \mathbb{C}^*$. If (s_0, s_4) is the point corresponding to $(X, \omega, C) \in M$, then (rs_0, rs_4) is the point corresponding to $(X, C, r^{-2}\omega)$.*

Proof. We know that the isomorphism $(x_0, x_1, x_2, x_3, x_4) \mapsto (x_0, rx_1, rx_2, rx_3, rx_4)$ between $X_{s_0,s_4} = X_{t_0,t_4}$ and $X_{rs_0,rs_4} = X_{r^2 t_0, r^{10} t_4}$ takes $\omega(t_0, t_4)$ to $r^{-2}\omega(r^2 t_0, r^{10} t_4)$ (see [56], sec. 2.1). Using that $t_0 = s_0^2$ and $t_4 = s_4^{10}$, we have our result. We just need to check that it maps the curve C to its correspondent. Indeed, we have

$$C_{rs_0,rs_4} = \left\{ r^2 s_4^2 x_0 + x_1 = 0, x_2 + x_3 = 0, rs_4 x_4^2 + rs_0 \sqrt{5} x_1 x_3 = 0 \right\} \subset X_{rs_0,rs_4}$$

[1] `www.impa.br/~felipe.espreafico/expansionsi`

and

$$C_{s_0,s_4} = \left\{ s_4^2 x_0 + x_1 = 0, x_2 + x_3 = 0, s_4 x_4^2 + s_0\sqrt{5} x_1 x_3 = 0 \right\} \subset X_{s_0,s_4}.$$

Taking a point of the second curve and applying the isomorphism, we have that the equations of the first one are satisfied.

$$\begin{aligned}
&r^2 s_4^2 x_0 + (r^2 x_1) = r^2(s_4^2 x_0 + x_1) = 0, \\
&(r^2 x_2) + (r^2 x_3) = r^2(x_2 + x_3) = 0, \\
&r s_4 (r^2 x_4)^2 + r s_0 \sqrt{5}(r^2 x_1)(r^2 x_3) = r^5(s_4 x_4^2 + s_0\sqrt{5} x_1 x_3) = 0.
\end{aligned}$$

This finishes the proof. □

Considering the zero locus of the function f given in (4.7) in the product $\mathbb{P}^4 \times \mathsf{S}_{\mathrm{op}}$, we get a family $\mathscr{X} \to \mathsf{S}_{\mathrm{op}}$. Now, $C_\pm$ induces a subfamily $\mathscr{Y} \subset \mathscr{X}$ corresponding to the curves. Using the definitions of the Gauss-Manin connection and Algebraic Relative Cohomology from chapter 2, we can already prove the main results. For this, as we said, we need the non-homogenous version of the Picard-Fuchs equation

$$\theta^4 - z\left(\theta + \frac{1}{5}\right)\left(\theta + \frac{2}{5}\right)\left(\theta + \frac{3}{5}\right)\left(\theta + \frac{4}{5}\right) = 15\frac{\sqrt{5^{-5} z}}{8} \tag{5.5}$$

satisfied by the integral of a holomorphic three-form on X_ψ over the homology class connecting the two curves $C_\pm$. Above, $z = \psi^{-5}$ and $\theta = z\frac{\partial}{\partial z}$. The equation in the form above is in [62, page 1170] using coordinates t with $z = 5^5 e^t$. Observe that if we consider the right-hand side of (5.5) to be zero, we get the classical equation appearing on [14] for which the periods given by integrals of the holomorphic three form over absolute homology classes are solutions.

Due to what we discussed in Chapters 2 and 3, there is a natural Mixed Hodge Structure in $H^3_{dR}(X,Y)$ and it interacts with the Gauss-Manin connection in a natural way, since the connection commutes with the long exact sequence (see Proposition 2.12). In our case, things are even better since we have some vanishing on certain cohomologies.

Proposition 5.5. *For $Y = C_+ \cup C_-$ and X a mirror quintic, the image of the map $\alpha : H^2_{\mathrm{dR}}(Y) \to H^3_{\mathrm{dR}}(X,Y)$ defined via the long exact sequence is one-dimensional and it is contained in $F^1 H^3_{\mathrm{dR}}(X,Y) \setminus F^2 H^3_{\mathrm{dR}}(X,Y)$, where F represents the Hodge filtration. Also, this image is exactly the second piece of the weight filtration $W_2(H^3_{\mathrm{dR}}(X,Y))$. In particular, any generator of this image satisfies properties (iii) and (iv) of Definition 5.1.*

Proof. Recall that, as Y is the union of two $\mathbb{P}^1$, it has $F^2 H^2_{\mathrm{dR}}(Y) = 0$ and $F^1 H^2_{\mathrm{dR}}(Y) = H^2_{\mathrm{dR}}(Y)$. Therefore, as the long exact sequence of the pair is a sequence of mixed Hodge structures, we conclude that the image is contained in F^1 and not in F^2 (since we have $\alpha(F^p(H^2_{\mathrm{dR}}(Y))) = \mathrm{Im}(\alpha) \cap F^p H^3_{\mathrm{dR}}(X,Y)$). For the part about the weight filtration, we just need to use that the weight filtration for $H^2_{\mathrm{dR}}(Y)$ is given by $W_0 = 0$, $W_1 = 0$ and $W_k = H^2_{\mathrm{dR}}(Y)$ for $k \geq 2$. Therefore, by again using that the exact sequence is a sequence of mixed Hodge structures, we conclude that the image of α is $W_2 H^3_{\mathrm{dR}}(X,Y)$. □

5.3 Proofs of the Main Theorems

In this section, our goal is to prove the two main theorems stated in Section 5.1. For simplicity, throughout this section, we keep denoting $Y = C_+ \cup C_-$.

5.3.1 Proof of Theorem 5.2

Consider the basis $W = \{\omega_1, \dots, \omega_4\}$ for $H^3_{\mathrm{dR}}(X)$, with ω_1 a holomorphic 3-form and $\omega_i := \nabla_{\frac{\partial}{\partial t_0}}(\omega_{i-1})$, where $t_0 := s_0^2$ and ∇ is the Gauss-Manin connection on the absolute cohomology (see Section 5.2). Notice that we use derivatives with respect to t_0 instead of s_0 since it makes it easier to compare with the absolute case. Of course, we can go from t_0 to s_0 via the relation

$$\frac{\partial}{\partial t_0} = \frac{1}{2s_0}\frac{\partial}{\partial s_0}.$$

Using the fact that the map $H^3_{dR}(X,Y) \to H^3_{dR}(X)$ is surjective, it is possible to choose elements $\omega_1, \dots, \omega_4 \in H^3_{dR}(X,Y)$ corresponding to the basis W. Then, we choose a generator of the image of the connecting morphism $\mathrm{Im}(\mathrm{H}^2_{\mathrm{dR}}(Y) \to \mathrm{H}^3_{\mathrm{dR}}(X,Y))$ and call it ω_0. This generator is chosen to be the image of the Poincaré dual of the difference of the homology classes $[C_+]$ and $[C_-]$. In this way, we get that the integral of ω_0 over the homology class connecting the two curves is 1. Now, consider the matrix:

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 \\ s_7 & c & s_6 & s_5 & 0 \\ s_8 & s_1 & s_2 & s_3 & d \end{bmatrix} \tag{5.6}$$

and assume that it is invertible, which implies $s_5 \neq 0$. The base $\alpha = S\omega$ satisfies all properties on Definition 5.1 except for (ii). Indeed, by Proposition 5.5 above, (iii) and (iv) are satisfied and, as the map $H^3(X,Y) \to H^3(X)$ preserves filtrations and the Gauss-Manin connection sends F^i to F^{i-1}, we have condition (i). Condition (v) is satisfied by construction. Demanding $S[\langle\omega_i,\omega_j\rangle]S^{\mathrm{tr}} = \Phi$, that is, condition (ii), we get equations relating a,b,c,d and the other variables:

$$cb - s_6 a = 3125 s_0^6 + s_2, \tag{5.7}$$

$$d = -b s_5, \tag{5.8}$$

$$s_5 a = -3125 s_0^8 - s_3, \tag{5.9}$$

$$d = 625\left(s_4^{10} - s_0^{10}\right). \tag{5.10}$$

To perform this computation, we make use of the intersection product computed in [56, Proposition 3] and the fact that α_0 is degenerate for the intersection product. These relations imply that we can drop the variables a, b, c and d and only consider, besides s_0 and s_4, five coordinates s_1, s_2, s_3, s_5, s_6, which are the same as the corresponding t_i in [56], and the extra two coordinates s_7 and s_8 which only appear in the relative case. Notice that, for each matrix S, we obtain a different basis α and for each basis α, we obtain a matrix by inverting S and solving $\omega = S^{-1}\alpha$.

5.3.2 Proof of Theorem 5.3

To prove Theorem 5.3, we need first to compute the Gauss-Manin connection in the basis α. Fix $z = \frac{t_4}{t_0^5} = \frac{s_4^{10}}{s_0^{10}}$ and consider the non-homogenous Picard-Fuchs differential equation (5.5). Let $\eta_1 = t_0\omega_1$ and $\eta_0 = \omega_0$. These forms are the ones we get by pulling back ω_0 and ω_1 via the isomorphism $X_{1,\frac{s_4}{s_0}} \cong X_{s_0,s_4}$ (see Proposition 5.4). Define $\eta_i = \nabla_{\frac{\partial}{\partial z}}(\eta_{i-1})$. By the definition of z, we get a relation between $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial t_0}$ given by

$$\frac{\partial}{\partial z} = \frac{-1}{5}\frac{t_0^6}{t_4}\frac{\partial}{\partial t_0}. \tag{5.11}$$

It is easy therefore to get a relationship between the basis η and ω. We call this matrix C. As we observed, the Picard-Fuchs equation (5.5) is satisfied by the integral of η_1 over the homology class connecting the curves C_+ and C_-. Using that $\int \eta_0 = \int \omega_0 = 1$, the fact that η_1 satisfy (5.5) implies the following equality after using the relationship between the Gauss-Manin connection and integrals (see Chapter2, Proposition 2.12):

$$\int_{\delta_0} \nabla_{\frac{\partial}{\partial z}} \eta_4 = \frac{-p}{z^4(z-1)} \int_{\delta_0} \eta_0 + \frac{-24}{625z^3(z-1)} \int_{\delta_0} \eta_1 + \\ + \frac{-24z+5}{5z^3(z-1)} \int_{\delta_0} \eta_2 + \frac{-72z+35}{5z^2(z-1)} \int_{\delta_0} \eta_3 + \frac{-8z+6}{z(z-1)} \int_{\delta_0} \eta_4. \quad (5.12)$$

By comparing the integrands, we can see that the Gauss-Manin matrix in the basis η is given by:

$$\mathsf{B}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ a_0 & a_1 & a_2 & a_3 & a_4 \end{bmatrix} dz, \quad (5.13)$$

where a_i are the coefficients of $\frac{\partial^i}{\partial z^i}$ in 5.5. To end, we compute

$$\mathsf{B}_2 := (dC + C \cdot \mathsf{B}_1)C^{-1}, \quad (5.14)$$

which is the Gauss-Manin connection written in basis ω. Observe that the submatrix formed by lines and columns from 2 to 5 is the Gauss-Manin connection in $H^3_{dR}(X)$ as computed in [56] Section 2.6. To compute the matrix in the basis α from Theorem 5.2, we compute:

$$\mathsf{A} = (dS + S \cdot \mathsf{B}_2)S^{-1}, \quad (5.15)$$

where S is given in (5.6).

Proof of Theorem 5.3. We take an unknown vector field R and let its first six coordinates be equal to the ones in [56, Theorem 3]. As the 4×4 submatrix of the Gauss-Manin Connection is the same as in the absolute case, by direct computation, we end up with

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{5^8(s_0^{10}-s_4^{10})^2}{s_5^3} s_7 & 0 & 0 & \frac{5^8(s_4^{10}-t_0^{10})^2}{s_5^3} & 0 \\ ds_7(\mathsf{R})+s_8 & 0 & 0 & 0 & -1 \\ ds_8(\mathsf{R})+\frac{5^{12}(s_0^{10}-s_4^{10})}{s_5} p & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.16)$$

after plugging R in the matrix A from (5.15). Now, recalling that $\mathrm{d}s_i(\mathsf{R})$ is the i-th coordinate of the vector field R and that the first column has to have only zeros except for the third line, we can determine the other coordinates of R uniquely. This gives us the desired vector field and ends the proof. □

5.4 Relationship with Periods

In this section, our goal is to explain why the functions Y and F appear. For this, we need to look at the period domain of the space T_{op} from Theorem 5.2. For us, a period is simply a number obtained by the integration of differential forms over cycles in homology. Here, we are especially interested in the integration of 3-forms over 3-dimensional cycles. Consider a symplectic basis of the homology group $H_3(X)$ given by $\{\delta_1, \delta_2, \delta_3, \delta_4\}$. Also, let δ_0 be the homology class connecting the two rational curves C_+ and C_-. Of course, these five homology classes form a basis for $H_3(X, Y)$.

Definition 5.6. *The period matrix is defined as*

$$\mathsf{P} = [p_{ij}] = \left[\int_{\delta_i} \alpha_j\right]_{ij}, \tag{5.17}$$

where the α_j form a basis satisfying the conditions given in Definition 5.1.

Using Poincaré duality, one can easily see that this matrix is related to the intersection matrix of the α_i's by the formula

$$[\langle \alpha_i, \alpha_j \rangle] = \left[\int_{\delta_i} \alpha_j\right]^{\mathrm{T}} \Psi^{-\mathrm{T}} \left[\int_{\delta_i} \alpha_j\right], \tag{5.18}$$

where Ψ is the intersection matrix of the basis δ, which is given by

$$\Psi := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

Definition 5.7. *Let* G *be the group given by:*

$$\mathsf{G} := \left\{ g = \begin{pmatrix} 1 & 0 & 0 & h_1 & h_2 \\ 0 & g_{11} & g_{12} & g_{13} & g_{14} \\ 0 & 0 & g_{22} & g_{23} & g_{24} \\ 0 & 0 & 0 & g_{33} & g_{34} \\ 0 & 0 & 0 & 0 & g_{44} \end{pmatrix}, \quad h_k, g_{ij} \in \mathbb{C} \quad \begin{matrix} g_{11}g_{44} = 1, \\ g_{22}g_{33} = 1, \\ g_{12}g_{44} + g_{22}g_{34} = 0, \\ g_{13}g_{44} + g_{23}g_{34} - g_{24}g_{33} = 0. \end{matrix} \right\} \tag{5.19}$$

The group G acts in an element (X, α) in the moduli space by the right as $(X, \alpha) \bullet g = (X, \alpha g)$ where α is seen as a line vector. Considering the relations, we can write this group in terms of six "g"-coordinates and two "h"-coordinates, as below:

$$(g_1, g_2, g_3, g_4, g_5, g_6, h_1, h_2) = \begin{pmatrix} 1 & 0 & 0 & h_1 & h_2 \\ 0 & g_1^{-2} & -g_3 g_1^{-1} & (-g_3 g_6 + g_4) g_1^{-2} & (-g_3 g_4 + g_5) g_1^{-2} \\ 0 & 0 & g_2^{-1} & g_6 g_2^{-1} & g_4 g_2^{-1} \\ 0 & 0 & 0 & g_2 & g_2 g_3 \\ 0 & 0 & 0 & 0 & g_1^2 \end{pmatrix}. \tag{5.20}$$

Notice that our coordinate g_1 is different from [56]: ours is the square root of the one in that article.

Proposition 5.8. *The action of* G *written on the coordinates s_i of* T_{op} *is*

$$\begin{aligned}
g \bullet s_0 &= s_0 g_1 \\
g \bullet s_1 &= s_1 g_1^2 + c g_1 g_2 g_3 + a g_1 g_2^{-1} g_4 - g_3 g_4 + g_5 \\
g \bullet s_2 &= s_2 g_1^3 + s_6 g_1^2 g_2 g_3 + b g_1^2 g_2^{-1} g_4 \\
g \bullet s_3 &= s_3 g_1^4 + s_5 g_1^3 g_2 g_3 \\
g \bullet s_4 &= s_4 g_1 \\
g \bullet s_5 &= s_5 g_1^3 g_2 \\
g \bullet s_6 &= s_6 g_1^2 g_2 + b g_1^2 g_2^{-1} g_6 \\
g \bullet s_7 &= s_7 g_2 + h_1 \\
g \bullet s_8 &= s_7 g_2 g_3 + s_8 g_1 + h_2
\end{aligned}$$

where a, b, c are the expressions given in terms of the coordinates s_i in (5.6).

Proof. We start with a pair (X_{s_0,s_4},ω_1). The form ω_1, together with its derivatives and a form ω_0 yields a basis of $H^3(X,Y)$. Multiplying by the matrix S from equation (5.6) we get a basis satisfying the conditions in Definition 5.1 depending on the coordinates s_i. Now, let $g \in G$ act. By definition, $\alpha_1 = \omega_1$ would be multiplied by g_1^{-2}. To write the new element of the moduli space in coordinates, we need to normalize ω_1. Consider the form $g_1^2\omega_1$ in the beginning. After this change, we need to multiply the basis ω by the matrix

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 \\ 0 & 0 & k^2 & 0 & 0 \\ 0 & 0 & 0 & k^3 & 0 \\ 0 & 0 & 0 & 0 & k^4 \end{pmatrix}$$

where $k = g_1^2$. This is because the other forms of the basis ω are derivatives of ω_1.

Notice that, by doing this, we are considering the point (ks_1, ks_4) of the moduli space of mirror quintics enhanced with a holomorphic 3-form and two rational curves. Now, the matrix $g^T SK$ takes the basis $\omega_0, \omega_1, \ldots, \omega_4$ to its image by the action of g. The entries of this matrix are the coordinates of the image. For example, the entry $(5,2)$ should be the coordinate t_1, etc. After completing this computation, we get the result. □

5.4.1 The τ-matrix

We want to consider the orbits of the action of G on T_{op} and their images by the period map. For this, we notice that G acts on the space of matrices by right-multiplication. This action preserves the relations (5.18) and is compatible with the action on T_{op}, in the sense that the period matrix relative to a basis $\alpha \bullet g$ is simply Ag, where A is the matrix with respect to α.

Proposition 5.9. *For any period matrix* P *satisfying the relations* (5.18), *there exists a unique* $g \in G$ *such that* $\mathsf{P}g$ *can be written in the form*

$$\tau = \begin{pmatrix} 1 & \tau_4 & \tau_5 & 0 & 0 \\ 0 & \tau_0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \tau_1 & \tau_3 & 1 & 0 \\ 0 & \tau_2 & -\tau_0\tau_3 + \tau_1 & -\tau_0 & 1 \end{pmatrix}, \tag{5.21}$$

for some τ_i.

Proof. Write g in the form (5.20). Multiplying g by a general matrix P and using the relations (5.18), we get a matrix of the form

$$\mathsf{P}g = \begin{pmatrix} 1 & * & * & * & * \\ 0 & * & 1 & 0 & * \\ 0 & 1 & 0 & 0 & 0 \\ 0 & * & * & * & * \\ 0 & * & * & * & * \end{pmatrix}.$$

Using the computations done in [56, Section 3.3], writing down the equations for $\mathsf{P}g = \tau$ when the entries in τ are independent of τ_i, we get that the first coordinates of $g' := g^{-1}$ are necessarily given by

$$
\begin{aligned}
(g'_1)^2 &= \mathsf{P}_{21}^{-1},\\
g'_2 &= \frac{-\mathsf{P}_{21}}{\mathsf{P}_{11}\mathsf{P}_{22}-\mathsf{P}_{12}\mathsf{P}_{21}},\\
g'_3 &= \frac{-\mathsf{P}_{22}}{\mathsf{P}_{21}},\\
g'_4 &= \frac{-\mathsf{P}_{12}\mathsf{P}_{23}+\mathsf{P}_{13}\mathsf{P}_{22}}{\mathsf{P}_{11}\mathsf{P}_{22}-\mathsf{P}_{12}\mathsf{P}_{21}},\\
g'_5 &= \frac{\mathsf{P}_{11}\mathsf{P}_{22}\mathsf{P}_{24}-\mathsf{P}_{12}\mathsf{P}_{21}\mathsf{P}_{24}+\mathsf{P}_{12}\mathsf{P}_{22}\mathsf{P}_{23}-\mathsf{P}_{13}\mathsf{P}_{22}^2}{\mathsf{P}_{11}\mathsf{P}_{21}\mathsf{P}_{22}-\mathsf{P}_{12}\mathsf{P}_{21}^2},\\
g'_6 &= \frac{\mathsf{P}_{11}\mathsf{P}_{23}-\mathsf{P}_{13}\mathsf{P}_{21}}{\mathsf{P}_{11}\mathsf{P}_{22}-\mathsf{P}_{12}\mathsf{P}_{21}}
\end{aligned}
$$

It suffices to compute h'_1 and h'_2. After computing $\mathsf{P}g = \mathsf{P}g'^{-1}$, we get:

$$
\tau = \begin{pmatrix}
1 & \frac{\mathsf{P}_{01}}{\mathsf{P}_{21}} & \frac{\mathsf{P}_{01}\mathsf{P}_{22}-\mathsf{P}_{02}\mathsf{P}_{21}}{\mathsf{P}_{11}\mathsf{P}_{22}-\mathsf{P}_{12}\mathsf{P}_{21}} & P & Q\\
0 & \frac{\mathsf{P}_{11}}{\mathsf{P}_{21}} & 1 & 0 & 0\\
0 & 1 & 0 & 0 & 0\\
0 & \frac{\mathsf{P}_{31}}{\mathsf{P}_{21}} & \frac{-\mathsf{P}_{21}\mathsf{P}_{32}+\mathsf{P}_{22}\mathsf{P}_{31}}{\mathsf{P}_{11}\mathsf{P}_{22}-\mathsf{P}_{12}\mathsf{P}_{21}} & 1 & 0\\
0 & \frac{\mathsf{P}_{41}}{\mathsf{P}_{21}} & \frac{-\mathsf{P}_{21}\mathsf{P}_{42}+\mathsf{P}_{22}\mathsf{P}_{41}}{\mathsf{P}_{11}\mathsf{P}_{22}-\mathsf{P}_{12}\mathsf{P}_{21}} & -\frac{\mathsf{P}_{11}}{\mathsf{P}_{21}} & 1
\end{pmatrix} \tag{5.22}
$$

where P and Q depend linearly on h'_1, h'_2. Making $P = Q = 0$, we find expressions for h_1 and h_2. This shows existence and uniqueness. □

Define L_{op} as the locus in the moduli space T_{op} from Theorem 5.2 for which the period matrix is of the form (5.21). Our goal is to express the functions s_i restricted to this locus in some coordinate. To do this, we first consider the points of T_{op} of the form $(1,0,0,0,y,1,0,0,0)$, where $y^{10} = z$ (the same coordinate used in Section 5.3.2). Then, we compute the period matrix P for these points and find the elements $g \in \mathsf{G}$ for which P is of the form (5.21). Then, by computing the elements $(1,0,0,0,y,1,0,0,0)\bullet g$, we will get expressions for the coordinates of L_{op}. We consider the periods

$$
x_{ij} = \int_{\delta_j} \tilde{\eta}_i,
$$

where $\tilde{\eta}_0 = \omega_0 = \alpha_0$, $\tilde{\eta_1}$ is the holomorphic three form associated to the point $(1, \frac{s_4}{s_0})$ of the moduli space of triples $(X, \omega, C_\pm)$ defined in Section 5.2 and $\tilde{\eta_i} = \theta(\tilde{\eta_{i-1}})$ (recall $\theta = z\frac{\partial}{\partial z}$). These periods are related to the solutions of the Picard-Fuchs equation via the matrix

$$
\begin{pmatrix} x_{01}\\ x_{11}\\ x_{21}\\ x_{31}\\ x_{41}\end{pmatrix}
=
\begin{pmatrix}
\frac{1}{2\pi^2} & 0 & 0 & \frac{5^4}{2\cdot(2\pi i)^2} & \frac{5^4}{4\cdot(2\pi i)^3}\\
0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 1\\
0 & 0 & 5 & \frac{5}{2} & -\frac{25}{12}\\
0 & -5 & 0 & -\frac{25}{12} & 200\frac{\zeta(3)}{(2\pi i)^3}
\end{pmatrix}
\begin{pmatrix}
\varphi\\
\frac{1}{5^4}\psi_3\\
\frac{2\pi i}{5^4}\psi_2\\
\frac{(2\pi i)^2}{5^4}\psi_1\\
\frac{(2\pi i)^3}{5^4}\psi_0
\end{pmatrix}, \tag{5.23}
$$

where ψ_i are solutions for the homogenous equation as it is presented in [56, Introduction] and φ is the solution for inhomogeneous equation (5.5) given by the series:

$$
2\sum_{m\,\mathrm{odd}}^{\infty} \frac{(5m)!!}{(m!!)^5}(5^{-5}z)^{m/2}
$$

where the double exclamation point means we multiply all the odd numbers less or equal to the number (e.g. $7!! = 1\cdot3\cdot5\cdot7 = 105$). The expression for x_{01} is taken from [54] and [62]. Notice that the notation for the series above in [54] is different: they take $\tau = \frac{\varphi}{30}$. The expressions for the other periods are taken from [56, Introduction]. To find the period matrix in terms of x_{ij}, we need to change from $\tilde{\eta}$ to the basis α from the moduli space. We consider, therefore, the matrices

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -5^5 & -5^4(z-1) & 0 & 0 \\ 0 & -\frac{5}{z-1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5^4(z-1) \end{pmatrix}$$

and

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & 0 & 0 \\ 0 & 2 & 15 & 25 & 0 \\ 0 & -6 & -55 & -150 & -125 \end{pmatrix},$$

where T takes the basis η to the basis ω and S takes ω to α. Notice that the matrices above have already been used in Section 5.3 but for general s_i. Therefore, the period matrix is simply $\mathsf{P} = [x_{ij}](ST)^T$. Now, using Proposition 5.9 and Proposition 5.8, it is easy to find $g \in \mathsf{G}$ for which $\mathsf{P}g$ is of the form (5.21) and compute the action of g on $(1,0,0,0,y,1,0,0,0)$ to get the s_i restricted to L_{op}. The first 7 were already computed in [56, Theorem 1] and the other two are given below:

$$s_7 = 5^7(z-1)\frac{x_{01}x_{12}x_{23} - x_{01}x_{13}x_{22} - x_{02}x_{11}x_{23} + x_{02}x_{13}x_{21} + x_{03}x_{11}x_{22} - x_{03}x_{12}x_{21}}{x_{21}}, \qquad (5.24)$$

$$s_8 = 5^7(z-1)(x_{01}x_{24} - x_{02}x_{23} + x_{03}x_{22} - x_{04}x_{21}) + \\ + 5^6 z\left(x_{01}x_{22} + \frac{5}{2}x_{01}x_{23} - x_{02}x_{21} - \frac{5}{2}x_{03}x_{21}\right). \qquad (5.25)$$

Proposition 5.10. *The Gauss-Manin connection restricted to the locus L can be computed in terms of the coordinate τ_0. It is given by:*

$$\mathsf{A}|_L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{d\tau_5}{d\tau_0} & 0 & 0 & \frac{d\tau_3}{d\tau_0} & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} d\tau_0, \qquad (5.26)$$

where τ is given by (5.22).

Proof. To prove this, use the fact that the Gauss-Manin connection commutes with integrals, in the following sense:

$$d\left(\int_\delta \omega\right) = \int_\delta \nabla\omega,$$

where the integration on the right-hand side takes place only on $\mathrm{H}^3_{\mathrm{dR}}(X,Y)$ (recall that $\nabla\omega$ can be written as the sum of elements of the form $\omega' \otimes s$, where s a form in Ω^1_T). Now, using this, we get:

$$d\tau = \left[\int_{\delta_i} \nabla\alpha_j\right]_{i,j} = \left[\sum_k \int_{\delta_i} a_{jk}\alpha_k\right]_{i,j} = \left[\sum_k a_{jk}\tau_{ik}\right]_{i,j} = \left[\sum_k \tau_{ik}a^T_{kj}\right]_{i,j} = \tau\cdot\mathsf{A}^T.$$

This implies that the Gauss-Manin connection has to be given by

$$\mathsf{A}|_L = d\tau^{\mathrm{T}} \cdot \tau^{-\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -\tau_5 d\tau_0 + d\tau_4 & 0 & d\tau_0 & -\tau_3 d\tau_0 + d\tau_1 & -\tau_1 d\tau_0 + \tau_0 d\tau_1 + d\tau_2 \\ d\tau_5 & 0 & 0 & d\tau_3 & -\tau_3 d\tau_0 + d\tau_1 \\ 0 & 0 & 0 & 0 & -d\tau_0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

By using Griffiths transversality and the fact that α_0 is on F^1, we conclude that positions (2,1), (2,4) and (3,5) have to be zero since our basis respects the Hodge filtration. After taking $d\tau_0$ out, we have the result. This yields the relations

$$\tau_1 = -\frac{d\tau_2}{d\tau_0} - \tau_0 \frac{d\tau_1}{d\tau_0}, \tag{5.27}$$

$$\tau_3 = \frac{d\tau_1}{d\tau_0}, \tag{5.28}$$

$$\tau_5 = \frac{d\tau_4}{d\tau_0}. \tag{5.29}$$

Notice the first two had already been computed in [56]. □

By the uniqueness statement from Theorem 5.3, we conclude that $\frac{\partial}{\partial \tau_0}$ is the vector field R restricted to the locus L_{op}. If we consider the coordinate $q = e^{2\pi i \tau_0}$, then R becomes $2\pi i q \frac{\partial}{\partial q}$. Writing the functions F and Y (or $\frac{d\tau_5}{d\tau_0}$ and $\frac{d\tau_3}{d\tau_0}$, respectively) in the coordinate q gives us virtual counts of disks with boundary on the real quintic and virtual counts of rational curves on a quintic threefold. To see this, we just need to look at the expressions of the τ_i in terms of x_{ij} we get, for example, that τ_4 is given by $\frac{x_{01}}{x_{21}}$, i.e., the quotient of a solution for the non-homogenous Picard-Fuchs equation by the holomorphic solution to the homogenous equation. This shows that the expressions for F and Y are the expressions in periods that we have for the disk potential and the Yukawa coupling.

To end, we list some of the possible extensions that can be made to our work. Firstly, it is natural to consider the case of two moving rational curves or even a family of divisors in the mirror quintic and see if we get a generalization of Jacobi forms. A nice starting point would be the paper [33] in which the authors consider families of divisors on the mirror quintic from a Physical point of view. Secondly, we could consider higher genus Gromov-Witten invariants and try to carry out a similar program. Finally, there are still some details missing in the realization of GMCD in general. A very interesting program would be to try to construct a more rigorous framework to define the moduli spaces and the functions we are working with.

Part II

Refinements of Donaldson-Thomas Invariants

Chapter 6

Donaldson-Thomas Invariants and their Motivic Refinements

6.1 The Grothendieck Ring of Varieties

Let k be a field of characteristic zero.

Definition 6.1. *The Grothendieck ring of varieties, denoted by $K_0(Var(k))$, is the abelian group generated by isomorphism classes of all varieties over k modulo relations*

$$[X] = [Y] + [X \setminus Y],$$

where Y is closed in X.

The product structure that makes $K_0(Var(k))$ a ring is given by the usual Cartesian product $[X]\cdot[Y] = [X \times Y]$. The class of the affine line $[\mathbb{A}^1_k]$ is denoted by $\mathbb{L}$ in $K_0(Var(k))$ and we set

$$\mathcal{M}_k := K_0(Var(k))[\mathbb{L}^{-\frac{1}{2}}],$$

which will be called ring of motivic weights.

Remark 6.2. *The following facts are true:*

- *If $f : X \to S$ is a Zariski locally trivial fibration with fiber F, then $[X] = [S]\cdot[F]$.*
- *If $f : X \to Y$ is a bijective morphism, it is true that $[X] = [Y]$.*

With this notation, the classes of cellular varieties can be computed in terms of $\mathbb{L}$, for example,

$$[\mathbb{P}^n] = 1 + \mathbb{L} + \mathbb{L}^2 + \cdots + \mathbb{L}^n$$

and a similar expression can be written for the Grassmanians:

$$[Gr(n,k)] = \frac{(\mathbb{L}^n - 1)(\mathbb{L}^{n-1} - 1)\cdots(\mathbb{L} - 1)}{(\mathbb{L}^{n-k})(\mathbb{L}^{n-k-1})\cdots(\mathbb{L}-1)\cdot(\mathbb{L}^k - 1)(\mathbb{L}^{k-1} - 1)\cdots(\mathbb{L} - 1)}.$$

Remark 6.3. *Notice that, over non-algebraically closed fields, besides the rational points, varieties have also points that correspond to Galois orbits over the algebraic closure. This gives rise to different classes of points in $K_0(Var(k))$, which are the classes of* Spec(L)*, with $L \supset k$.*

Example 6.4. *Over $\mathbb{R}$, we have two classes of points:* $[\mathrm{Spec}\mathbb{R}]$ *and* $[\mathrm{Spec}\mathbb{C}]$. *Therefore, although $\mathbb{P}^1(\mathbb{R})$ is a circle, there are non-rational points corresponding to pairs of complex conjugate points. Indeed, $\mathbb{P}^1$ is not isomorphic to $X = \{x^2 + y^2 = 1\} \subset \mathbb{A}^2$, since there are two complex conjugate points "at infinity". In terms of the Grothendieck ring of varieties, we get $[X] = \mathbb{L} + 1 - [\mathrm{Spec}\mathbb{C}] \in K_0(Var(\mathbb{R}))$.*

Over $k = \mathbb{C}$, there is a morphism to $\mathbb{Z}$ given by the Euler characteristic with compact support.

$$\chi_c : K_0(Var(\mathbb{C})) \to \mathbb{Z},$$

which can be extended to a morphism

$$\chi_c^{\mathbb{C}} : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z},$$

if we send $\mathbb{L}^{\frac{1}{2}}$ to -1 (since $\mathbb{L}$ goes to 1). The same can be done for considering the compactly supported Euler characteristic of the real points. In Chapter 7, we will introduce the $\mathbb{A}^1$-Euler characteristic, which is defined for varieties over any field. Notice that we have to be careful when extending to $\mathcal{M}_{\mathbb{R}}$. As $\chi_c^{\mathbb{R}}(\mathbb{L}) = -1$, the extended Euler characteristic takes values in $\mathbb{Z}[i]$, where $i = \sqrt{-1}$:

$$\chi_c^{\mathbb{R}} : \mathcal{M}_{\mathbb{R}} \to \mathbb{Z}[i].$$

If one does not want to consider compactly supported Euler characteristics, one can actually only consider projective varieties.

Proposition 6.5 ([12][Theorem 3.1)**.** *] Consider the group generated by projective varieties and the relations*

$$[X] - [Y] = [\mathrm{Bl}_Y X] - [E], \tag{6.1}$$

$$[\varnothing] = 0, \tag{6.2}$$

where $Y \subset X$ and $\mathrm{Bl}_Y X$ is the blow up of X along Y and E is the exceptional divisor. Then this group is isomorphic to $K_0(Var(k))$.

6.2 Donaldson-Thomas invariants

We consider Donaldson-Thomas invariants following [50]. These are simply numbers[1] obtained by integration against virtual classes of moduli spaces of ideal sheaves in the complex setting. If M is a smooth Calabi-Yau threefold, any ideal sheaf defines a subscheme N. The moduli spaces that we are interested in are the ones of the form $X = I_n(M,\beta)$ and consist of sheaves for which N represents the homology class β and $\chi(\mathcal{O}_N) = n$. Usually, we are interested in the cases for which N has dimension at most one, which implies that β is in $H_2(M)$. $I_n(M,\beta)$ is, then, isomorphic to a Hilbert Scheme of curves in M. In particular, for the case $\beta = 0$, we are considering exactly the situation in which N has dimension 0, that is, $I_n(M,0)$ is the Hilbert Scheme of points of M.

The DT invariants of M are defined by integrals:

$$DT(n,\beta) = \int_{[I_n(M,\beta)]^{vir}} 1 \tag{6.3}$$

where $[I_n(M,\beta)]^{vir}$ is a virtual class in the homology defined via obstruction theory. This is constructed in the same way as virtual classes for Gromov-Witten theory as in Chapter 1. It was proved, in [10], to depend only on the scheme structure of the moduli space and not on the chosen obstruction theory. In particular, if the moduli space ends up being smooth, this virtual class is, up to a factor $(-1)^{\dim I_n(M,\beta)}$, simply the Poincaré dual of the top Chern class, which means that the integral above is given, up to sign, by the Euler characteristic with compact support of the moduli space.

[1]For emphasis, one sometimes refers to them as "numerical DT-invariants", to distinguish them from their "motivic" version, defined afterward.

A natural way of refining such invariants is, therefore, to consider these virtual classes as elements in the Grothendieck ring of varieties, since the Euler characteristic gives us a natural morphism from this ring to $\mathbb{Z}$. If the moduli space is smooth these virtual classes should be given by the classes of $I_n(M,\beta)$ in the ring $\mathscr{M}_{\mathbb{C}}$, since the DT invariants coincide with the Euler characteristic. Notice, though, that, if the moduli space is singular, it is usually not straightforward to define such virtual classes in the Grothendieck ring of varieties. Pictorially, we want a commutative diagram as below:

$$\begin{tikzcd}
\{\text{Moduli Spaces } I_n(M,\beta)\} \arrow[dd, dashed, "\text{Virtual Classes}"'] \arrow[dr, "\text{DT Theory}"] & \\
& \mathbb{Z} \\
\mathscr{M}_{\mathbb{C}} \arrow[ur, "\chi_c"'] &
\end{tikzcd} \tag{6.4}$$

The idea of refining DT invariants via the Grothendieck ring of varieties was first considered in [40]. For us, the interest behind this is that, as already pointed out in the introduction and in [31], by considering real and complex Euler characteristics, we can, in some cases, recover real DT invariants from the same formulas in the Grothendieck ring. In general, the virtual classes are not in $\mathscr{M}_k$ (or $\mathscr{M}_{\mathbb{C}}$), but in a slightly bigger equivariant version of this ring. In the next section, we present a way to define virtual classes over a general field k of characteristic zero when the moduli spaces are defined over k and can be represented as a critical locus of a regular function, based mainly in the work of Denef and Loser [21]. In practice, this is the case in many interesting examples in which $D^b(X) \cong D^b(\mathbf{Q}-\mathrm{mod})$, that is, in which we can represent the moduli spaces as quiver varieties.

6.3 The Motivic Nearby Class

Let X be a smooth variety and $f : X \to \mathbb{A}^1$ be a regular function. Denote the central fibre $f^{-1}(0)$ by X_0. Our goal is to define a class $[Z]_{vir}$ for $Z = \{df = 0\}$, the critical locus of f, which encodes also the scheme structure of Z.

Definition 6.6. *Let μ_n be the group of $n-$roots of unity in $\bar{k}$. Notice that it has a structure of algebraic variety over k. We have maps $\mu_{nd} \to \mu_n$ given by $x \mapsto x^d$. This gives us a projective system. We denote the limit by $\hat{\mu}$. A good μ_n action on X is a group action $\mu_n \times X \to X$ which is a morphism of varieties such that each orbit is contained in an affine subvariety of X. A good $\hat{\mu}$-action is a group action $\hat{\mu} \times X \to X$ which factors through a good action of μ_n for some n.*

Definition 6.7. *Consider the abelian group generated by symbols $[X,\hat{\mu}] = [X]$, where X is a variety with a good $\hat{\mu}$-action, modulo isomorphisms compatible with the action. The relations are the same scissor relations $[X] = [Y] + [X \setminus Y]$ that we have for $K_0(Var(k))$ but such that the action on Y is induced by the one in X. Finally, there is one more relation given by $[X \times V] = [X \times \mathbb{A}^n]$, where V is the affine space of dimension n with a good $\hat{\mu}$-action and, on the other side, $\mathbb{A}^n$ has the trivial action. The product is given by the usual Cartesian product with the product action. The class of the affine line with the trivial action is denoted by $\mathbb{L}$. This ring will be denoted $K_0(Var^{\hat{\mu}}(k))$ and its localization $K_0(Var^{\hat{\mu}}(k))[\mathbb{L}^{-\frac{1}{2}}]$ by $\mathscr{M}_k^{\hat{\mu}}$.*

Take a resolution of $f : X \to \mathbb{A}^1$, that is, a map $h : Y \to X$, with Y smooth and irreducible such that $Y_0 = h^{-1}(X_0)$ has only normal crossings and the restriction $h : Y - Y_0 \to X - X_0$ is an isomorphism. We fix the notation:

- E_i, $i \in J$ denote the components of Y_0 and N_i their multiplicities;
- E_I, $I \subset J$ denote the intersections $\bigcap_{i \in I} E_i$;
- E_I°, $I \subset J$ denote $E_I - \bigcup_{j \notin I} E_j \cap E_I$

To take multiplicities into account, we define natural covering spaces $\tilde{E}_I^\circ \to E_I^\circ$, which are unramified and Galois with Galois groups given by μ_{m_I}, where m_I is the greatest common divisor of the multiplicities of all $E_i, i \in I$. Take $U \subset Y$ an open set such that $f \circ h = uv^{m_I}$, where u is a unit in $\mathscr{O}_Y(U)$ and v is a morphism $v : U \to k$. We can define $\tilde{E}_I^\circ$ by gluing the sets

$$\{(x,t) \in (U \cap E_I^\circ) \times \mathbb{A}^1 | t^{m_I} = u^{-1}(x)\}.$$

The considerations above show that $\tilde{E}_I^\circ$ is a Galois covering and that there exists a natural good action of μ_n (and thus, of $\hat{\mu}$) on it.

Definition 6.8. *The motivic nearby class of f is defined as follows:*

$$S_f := \sum_{\varnothing \neq I} (1 - \mathbb{L})^{|I|-1} [\tilde{E}_I^o] \in \mathscr{M}_k^{\hat{\mu}} \tag{6.5}$$

Among other remarks, the belief is stated in [21] that this is a motivic incarnation of the complex of nearby cycles of X_0, which was defined in Exposé XIII of [18]. Restricting S_f to a point $x \in X_0$, we get the local version, which the authors believe to be a motivic version of the classical Milnor fibre of f over x.

$$S_{f,x} := \sum_{\varnothing \neq I} (1 - \mathbb{L})^{|I|-1} [\tilde{F}_I^o] \in \mathscr{M}_k^{\hat{\mu}} \tag{6.6}$$

where F_i is the fiber of E_i over x.

Example 6.9. *Let $f : \mathbb{A}^2 \to \mathbb{A}^1$ be given by $f(x,y) = x^2 - y^2$. Then X_0 has already only normal crossings, which are the two lines E_1 and E_2 intersecting at the origin. We have:*

$$\begin{aligned} E_1^\circ &= E_1 - 0 \\ E_2^\circ &= E_2 - 0 \\ E_{1,2}^\circ &= 0 \end{aligned}$$

As all multiplicities are equal to 1, we do not need to worry about the covering spaces. Computing S_f, we get:

$$S_f = [E_1 - 0] + [E_2 - 0] + (1 - \mathbb{L})[0] = \mathbb{L} - 1 + \mathbb{L} - 1 + 1 - \mathbb{L} = \mathbb{L} - 1 \tag{6.7}$$

For $S_{f,0}$, we can write:

$$S_{f,x} = [(E_1 - 0) \cap h^{-1}(0)] + [(E_2 - 0) \cap h^{-1}(0)] + (1 - \mathbb{L})[0 \cap h^{-1}(0)] = 1 - \mathbb{L} \tag{6.8}$$

It is instructive to compare these results with what happens over $\mathbb{R}$ and $\mathbb{C}$. The Milnor fibre over a point $x \in X_0$, over $\mathbb{C}$, is defined as the intersection of the preimage $f^{-1}(w)$, for small $w \in \mathbb{C}$, with a small ball around x. This gives us a cylinder around x, which has Euler characteristic 0: this is what we get by making $\mathbb{L} = 1$ in equation 6.8. Considering the same definition over $\mathbb{R}$ (which cannot be done in all cases, considering that the topology of the Fiber

$f^{-1}(w)$ can change depending on w), we get two hyperbola segments. After intersecting with a ball, we get compact segments and the Euler characteristic should be 2. This is compatible with making $\mathbb{L} = -1$ in equation 6.8. If we do not intersect with the ball (that is, consider, in some sense, a global fibre), we would get an infinite cylinder ($\chi_c = 0$) or two non-compact branches of a hyperbola($\chi_c = -2$). This is compatible with equation 6.7.

As we will point out, $\mathbb{A}^1$-enumerative geometry has an arithmetic incarnation of the Milnor number for isolated singularities, which is the so-called "EKL class" of the gradient of f. Although, in order to compute DT invariants, we use the definition above in the case of non-isolated singularities, an interesting problem would be to understand the relationship between Milnor numbers and the class S_f. We elaborate on this in Section 8.2 of Chapter 8.

6.4 Virtual class of a critical locus

Definition 6.10. *Let X be a smooth variety and $f : X \to \mathbb{A}^1$ be a regular function. If Z is the critical locus of f and $X_0 = f^{-1}(0)$, the virtual class of the critical locus Z is defined as*

$$[Z]_{vir} = -\mathbb{L}^{-\frac{\dim X}{2}}(S_f - [X_0]) \in \mathscr{M}_k^{\hat{\mu}} \tag{6.9}$$

The idea behind this definition is that, over the smooth points of X_0, the Milnor fibre should correspond to X_0 itself. This means that this virtual class encodes exactly what is happening on the singular locus of X_0. Notice that this definition depends on the function f chosen, that is, it depends on how the variety is presented as a critical locus.

Remark 6.11. *Definition 6.10 was given in the paper [11]. The extension by the square root is related to the existence of a sign that appears when relating DT invariants to the Euler characteristic (see [10]).*

Example 6.12. *If $f = 0$ is the zero function, then $Z = X_0 = X$ (every point is critical). We can compute $S_f = 0$ using the fact that X_0 can be seen as the zero divisor. Therefore:*

$$[Z]_{vir} = -\mathbb{L}^{-\frac{\dim X}{2}}(0 - [X_0]) = \mathbb{L}^{-\frac{\dim X}{2}}[X]$$

This example shows that the virtual class of a smooth variety is its class in $\mathscr{M}_k$ up to multiplication by a factor. In terms of DT invariants, this factor corresponds to the sign difference that we get compared to the Euler characteristics.

The virtual class is not always in $\mathscr{M}_k$ but, over $\mathbb{C}$, Behrend, Bryan and Szendröi showed, in [11], that, when f is equivariant with respect to a torus action on X, $[Z]_{vir} \in \mathscr{M}_k$ and can be computed from the difference between the zero fiber and the generic fiber.

Proposition 6.13 (cf. Thm. B.1 in [11])**.** *Let $f : X \to \mathbb{C}$ be a regular morphism on a smooth quasi-projective variety. Let Z be the critical locus of f. Assume that there exists an action of a connected complex torus on X such that f is equivariant with respect to a primitive character. If there is a one-parameter subgroup $\mathbb{C}^\times \subset T$ such that the induced action is circle compact, that is, the set of fixed points is compact and the limit $\lim_{\lambda\to 0} \lambda x$ exists for all $x \in X$, then, the virtual class $[Z]_{vir}$ is given by*

$$[Z]_{vir} = -\mathbb{L}^{-\frac{\dim X}{2}}([X_1] - [X_0]) \in \mathscr{M}_{\mathbb{C}},$$

where $[X_1]$ is the class of the fibre $f^{-1}(1)$.

The result above strongly uses the fact that the varieties are defined over $\mathbb{C}$, but our computations in section 4 show that, at least in the case of degree zero invariants of $\mathbb{A}^3$, the real DT invariants are also encoded in the formula. Inspired by the work of Levine on localization formulas in $\mathbb{A}^1$-enumerative geometry [45], one may think that it could be possible to extend these arguments to more general fields.

Chapter 7

$\mathbb{A}^1$-Enumerative Geometry

$\mathbb{A}^1$-Enumerative geometry consists in applying the machinery of so-called $\mathbb{A}^1$-homotopy theory to classical enumerative problems. This machinery was introduced in the 90s, in the work of Morel and Voevodsky [53]. The main idea is to consider the affine line $\mathbb{A}^1$ rather than the interval $[0,1] \subset \mathbb{R}$ as a parameter space and thereby construct a homotopy theory for schemes. This allows us to define an algebraic or $\mathbb{A}^1$-degree for maps between schemes, which naturally generalizes the classical degree from algebraic topology. While the natural ring where this degree is defined is not $\mathbb{Z}$ but rather the Grothendieck-Witt ring of quadratic forms, the theory has the highly attractive feature of being defined over arbitrary fields, rather than just the complex or real numbers. In the next sections, we are going to define a refined version of the degree of a map and a refined version of the Chow groups, which will allow us to consider intersection theory as in classical enumerative geometry.

7.1 Quadratic Forms and the $\mathbb{A}^1$-degree

In this section, our goal is to introduce the Grothendieck-Witt ring of quadratic forms, denoted by $GW(k)$, which will allow us to make enumerative geometry over any field, as described above.

Definition 7.1. *The Grothendieck-Witt ring is the group completion of the set of all quadratic forms over k up to isometries with the operations*

$$q+q' : V \oplus W \to k \qquad q+q'(x,y) = q(x)+q'(y) \tag{7.1}$$

$$qq' : V \otimes W \to k \qquad qq'(x \otimes y) = q(x)q'(y) \tag{7.2}$$

where $q : V \to k$ and $q' : W \to k$ are quadratic forms representing isometry classes.

There are two important natural maps from $GW(k)$ to $\mathbb{Z}$ which are the rank and the signature.

- $\mathrm{rk} : \mathrm{GW(k)} \to \mathbb{Z}$ computes the dimension of the vector space in which the quadratic form is defined.

- $\mathrm{sgn} : \mathrm{GW(k)} \to \mathbb{Z}$ computes the signature of the quadratic form.

- If we have a field extension $k \subset L$, we can define a map $GW(L) \to GW(k)$ by simply considering composition with the trace map $Tr_{L/k} : L \to k$.

Remark 7.2. *Since every quadratic form can be diagonalized, $GW(k)$ is generated by elements of rank 1. These are represented by forms $q(x) = ax^2$, with $a \in k^\times$. They are denoted by $\langle a \rangle \in GW(k)$. If a is a square, $\langle a \rangle = \langle 1 \rangle$ in $GW(k)$, and therefore the generators of $GW(k)$ are the the elements in $k^\times/(k^\times)^2$. Using this notation, we write $\mathbb{H} = \langle 1 \rangle + \langle -1 \rangle$. This is called the hyperbolic form. It has the property that $\langle a \rangle \mathbb{H} = \mathbb{H}$ for any $a \in k$.*

The Grothendieck-Witt ring is suitable to refine the classical $\mathbb{Z}$-valued enumerative invariants (counts) defined over algebraically closed fields. Very roughly, the intuition is that we count points considering local "orientations" (as it is done over $\mathbb{R}$), which correspond to quadratic forms. Counting with such orientations allows us to get invariant counts over any field k. The ranks of these quadratic forms correspond to the size of the Galois orbit of the point and, therefore, they recover the counts over $\bar{k}$. The signature recovers the signed counts over $\mathbb{R}$ (when $k \subset \mathbb{R}$) since the two square classes over $\mathbb{R}$ will correspond to the two possible local orientations (signs). A good example is the definition of the $\mathbb{A}^1$-degree of a map from $\mathbb{A}^n \to \mathbb{A}^n$.

Definition 7.3. *Let $P : \mathbb{A}^n \to \mathbb{A}^n$ be a morphism and let x be a closed point with rational image $y = P(x)$. Assume P is étale at x and that x is isolated in its fiber. Then the local degree of P at x is given by:*

$$\deg_x^{\mathbb{A}^1}(P) = \mathrm{Tr}_{k(x)/k}\langle \det J(x) \rangle \tag{7.3}$$

where $J(x)$ denotes the derivative (Jacobian) of P at the point x and $k(x)$ denotes the residue field of x. The degree of P is simply the sum over all the preimages, which does not depend on y [35].

$$\deg^{\mathbb{A}^1}(P) = \sum_{x \in P^{-1}(y)} \mathrm{Tr}_{k(x)/k}\langle \det J(x) \rangle \tag{7.4}$$

Remark 7.4. *When P is not étale, the degree can still be defined using more involved machinery from homotopy theory. We refer the reader to [35, Section 2] for a more complete exposition and to [53] for technical details.*

Equation (7.3) corresponds to our local "orientation" and, in (7.4), we "count" all the points. The simplest example is the function $P(x) = x^2$ from $\mathbb{A}^1$ to $\mathbb{A}^1$.

Example 7.5. *Let $k = \mathbb{Q}$ and $P(x) = x^2$ in $\mathbb{A}^1$. The derivative is given by $J(x) = 2x$. Therefore, using equation (7.4) we can write, choosing $y = 1$ and $y = -1$, the following formulas.*

$$\deg^{\mathbb{A}^1}(P) = \mathrm{Tr}_{\mathbb{Q}(1)/\mathbb{Q}}\langle 2 \rangle + \mathrm{Tr}_{\mathbb{Q}(-1)/\mathbb{Q}}\langle -2 \rangle = \langle 2 \rangle + \langle -2 \rangle = \mathbb{H}$$

$$\deg^{\mathbb{A}^1}(P) = \mathrm{Tr}_{\mathbb{Q}(\sqrt{-1})/\mathbb{Q}}\langle 2\sqrt{-1} \rangle = \mathrm{Tr}_{\mathbb{Q}(\sqrt{-1})/\mathbb{Q}}\langle 1 \rangle = \mathbb{H}$$

In the second formula we used that $2\sqrt{-1} = (1+\sqrt{-1})^2$, and therefore its class in $GW(\mathbb{Q}(\sqrt{-1}))$ is $\langle 1 \rangle$. This corresponds to the simple quadratic form $x \mapsto x^2$, which, after composition with the trace, corresponds to $a + b\sqrt{-1} \mapsto 2a^2 - 2b^2$ which is $\mathbb{H}$.

Notice that $\mathrm{rk}(\mathbb{H}) = 2$, *which is the number of preimages in the algebraic closure, and that* $\mathrm{sgn}(\mathbb{H}) = 0$, *which is the real topological degree.*

7.2 $\mathscr{K}$ sheaves and Chow groups

7.2.1 Milnor $\mathscr{K}$-sheaves and classical Chow groups

Classical enumerative geometry lies on computing intersection products in the Chow groups. Our first goal in this section is to give a sheaf theoretic definition of the Chow cohomology groups. After this, we will be able to generalize this idea to give a definition of "refined" Chow cohomology groups, where we are going to be able to solve enumerative problems in terms of quadratic forms, as we will see in the next section. The results we present in this subsection, before going to the refined version, are from the paper [51].

Definition 7.6. *Let R be a ring. We define the algebra K_*R to be the quotient of the tensor algebra (over $\mathbb{Z}$) of the multiplicative group $R^\times$ by the ideal generated by elements of the form $[u]\otimes[1-u]$, for $u, 1-u \in R^\times$. These relations are known as Steinberg relations.*

In order to avoid confusion between the operations in K_*R and in R, we introduce the notation $[u]\in K_*$ for $u \in R^\times$. Then, we can write $[uv]=[u]+[v]$. Besides that, we will also drop the tensor product symbol when denoting multiplication.

Theorem 7.7. *Let F be a field. The algebra K_*F satisfies the following properties*

1. $[u][-u]=0$,
2. $\alpha\beta=(-1)^{mn}\beta\alpha$, *where $\alpha\in K_nF$ and $\beta\in K_mF$,.*
3. $[u]^2=[u][-1]$.

Proof. For the first part, we use that $-u(1-u^{-1})=1-u$. This implies:

$$[-u]+[1-u^{-1}]=[1-u]. \tag{7.5}$$

After multiplying both sides by $[u]=-[u^{-1}]$, we get, by definition:

$$[u][-u]=0. \tag{7.6}$$

For the second part, it suffices to show the fact for $n=m=1$, since degree 1 elements generate the algebra. Using the first part, we can write, for $u,v\in F^\times$:

$$\begin{aligned}[u][v]+[v][u]&=[u][v]+[u][-u]+[v][-v]+[v][u]=[u]([v]+[-u])+[v]([-v]+[u])=\\&=[u][-vu]+[v][-vu]=[uv][-vu]=0.\end{aligned} \tag{7.7}$$

The third assertion is also a consequence of the first one:

$$[u][u]=[u]([-1]+[-u])=[u][-1]. \tag{7.8}$$

□

Consider a discrete valuation v on the field F. Let its residue class field be denoted by k. Define R to be the associated DVR given by $v\geq 0$. Let U be the group of units of R, that is, elements for which $v=0$. Let ϕ be the natural quotient map $\phi: U\to k^\times$.

Lemma 7.8. *There exists a unique morphism ∂ from K_nF to $K_{n-1}k$ for which*

$$\partial([t][u_2]\dots[u_n])=[\phi(u_2)]\dots[\phi(u_n)],$$

where t is a generator of the maximal ideal of R and u_i's are units in U.

Proof. For $n=1$, we simply define ∂ to be the valuation map from $K_1F\cong F^\times$ to $\mathbb{Z}=K_0k$. Indeed, it takes $[t]$ to 1, for t any generator of the maximal ideal. It is a morphism since the valuation sends products to sums. Of course, any other morphism sending all generators of the maximal ideal to 1 has to send units to zero. As any element of $F^\times$ can be written as ut^r for u a unit, the map is uniquely determined in all elements of $F^\times$.

To finish, we just need to show that any element in K_nF can be written in terms of elements of the form $[t][u_2]\dots[u_n]$. Indeed, if we have an element of the form $\alpha=[u_1t^{r_1}]\dots[u_nt^{r_n}]$, we can write $\alpha = ([u_1]+r_1[t])...([u_n]+r_n[t])$. Using part 3 of Theorem 7.7, we have that $[t]^\ell = [t][-1]^{\ell-1}$ and we conclude that α is a sum of elements of the form $[t][v_2]\dots[v_n]$ and an element $[u_1]\dots[u_n]$. Now, in order not to depend on the choice of t, products of units have to be mapped to zero, since, for α a product of units, we have $[t]\alpha = [ut]\alpha = [u]\alpha + [t]\alpha$, which implies $[u]\alpha = 0$. Now, we can simply choose a generator t, define the map for $[t][u_2]\dots[u_n]$ and extend by linearity, sending $[u_1]\dots[u_n]$ to zero. Note that this is well-defined since if two elements have sum 1, they will still have sum 1 in the quotient. This proves uniqueness and existence. □

Remark 7.9. *The maps defined in Lemma 7.8 are surjective. This is a direct consequence of the fact that the quotient map $R \to k$ is surjective.*

Definition 7.10. *Let X be a smooth scheme over a field k. The nth Milnor K sheaf, denoted by $\mathcal{K}_n^M$ has its stalks given by, at each point $x \in X$, the ring $K_n(\mathcal{O}_{X,x})$. $\mathcal{K}_1^M = \mathcal{O}_X^\times$ and $\mathcal{K}_0^M$ is the constant sheaf $\mathbb{Z}$. Formally, we can define $\mathcal{K}_n^m$ to simply be the n-th part of the tensor algebra sheaf of $\mathcal{O}_X^\times$ and take the quotient by the Steinberg relations as in Definition 7.6.*

The next result shows how K-sheaves can be used to define the Chow groups in a cohomological way. This will be important for us in the definition of refined Chow groups. The idea will be to replace the Milnor K sheaves by Milnor Witt K sheaves and get a new version of Chow groups.Although this result is proved in [1], we present a proof of this theorem for completion, since the ideas present in it are important for understanding the analogous construction that is done in the $\mathbb{A}^1$ case.

Theorem 7.11 (Bloch-Kato)**.** *Let X be a smooth variety over k. Then, the Chow groups can be computed as sheaf-cohomology groups with coefficients on $\mathcal{K}_n^M$.*

$$H^n(X, \mathcal{K}_n^M) \cong CH^n(X). \tag{7.9}$$

Remark 7.12. *For $n = 1$, notice that Theorem 7.11 above is simply the classical identity:*

$$H^1(X, \mathcal{O}_X^\times) \cong \mathrm{Pic}(X) \cong CH^1(X).$$

Proof of Theorem 7.11. The idea is to consider a resolution of $\mathcal{K}_n^M$ by skyscraper sheaves of the following form:

$$0 \to \mathcal{K}_n^M \to \bigoplus_{x\in X^{(0)}} K_n(k(x)) \to \bigoplus_{x\in X^{(1)}} K_{n-1}(k(x)) \to \cdots \to \bigoplus_{x\in X^{(0)}} K_0(k(x)) \to 0, \tag{7.10}$$

where $X^{(i)}$ is the set of points of codimension i and the first map is the inclusion of algebras induced by the inclusion of local sections in the field of rational functions and the other maps are induced by the map ∂ from Lemma 7.8 (recall that if x is a codimension r point and y is a codimension $r+1$ point inside $\overline{\{x\}}$, the local ring of y can be seen inside $k(x)$, since any function that vanishes on x will also vanish on y: in other words, at an affine chart, we can identify x and y with two prime ideals of a ring A such that $I_x \subset I_y$ and therefore localizing outside I_y also localizes outside I_x).

First, we show that $\partial^2 = 0$. Take the r-th term of the sequence.

$$\cdots \to \bigoplus_{x\in X^{(r-1)}} K_\ell(k(x)) \to \bigoplus_{x\in X^{(r)}} K_{\ell-1}(k(x)) \to \bigoplus_{x\in X^{(r+1)}} K_{\ell-2}(k(x)) \to \dots \tag{7.11}$$

Choose any element $\alpha = [u_1]\dots[u_\ell] \in K_\ell(k(x))$ for some x of codimension $r-1$. For each point y of codimension r that is contained in the closure of x, choose a generator t_y of the local ring of y inside $k(x)$. The image of α in the component of the sum $\oplus_{x\in X^{(r)}} K_{\ell-1}(k(x))$ corresponding to y is simply 0, if no u_i lies on the ideal generated by t_y on the corresponding local ring, and it is $\pm[-1]^{i-1}[u_{z_1}]\dots[u_{z_{\ell-i}}]$ if there are i elements on the ideal generated by t_y, where the sign depends on the position of each element on the original sum: for example, if there are only on element on the ideal, the sign would be its index minus 1. To finally show that $\partial^2 = 0$, let z be a point of codimension $r+1$. Take all points of codimension r for which z belongs to the closure. Applying ∂, we will get a sum of elements of the form above, again considering for each y, a generator t_{zy} of the maximal ideal of the local ring corresponding to z inside $k(y)$. Notice that, the image consists of the same element in $K_{\ell-2}(k(z))$ since we are simply considering the prime ideal generated by t_y and t_{yz} in $k(x)$ and applying ∂. The difference is in the signs. Since we are considering all possible y, in the end, the signs cancel off and we get $\partial^2 = 0$.

Now, it's pretty straightforward to see that the sequence is indeed exact on the stalks. For this, take any point x. The stalks are direct sums over the points y such that $x \in \overline{y}$. Looking again at equation 7.11, if an element in $K_{\ell-1}(k(y))$, for some y, goes to zero by ∂, it means that this element is a product of units for all z of codimension 1 on $\overline{y}$. This implies that the elements are actual units of $\mathscr{O}_{\overline{x},y}$. This means that it is the image of an element from $K_\ell(k(x))$ by just adding a generator at the beginning.

As skyscraper sheaves are all flasque, we can compute cohomology by taking global sections. To compute the n-th cohomology group, we can look at the last terms:

$$\begin{array}{ccccccc} \dots & \to & \bigoplus_{x\in X^{(n-1)}} K_1(k(x)) & \to & \bigoplus_{x\in X^{(n)}} K_0(k(x)) & \to & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \\ \dots & \to & \bigoplus_{x\in X^{(n-1)}} k(x)^\times & \to & \bigoplus_{x\in X^{(n)}} \mathbb{Z} & \to & 0 \end{array}$$

If we recall that the boundary map, in this case, is simply the divisor map, associating a rational function on $\overline{\{x\}}$ to the corresponding principal divisor (since this is done exactly by computing valuations corresponding to each local ring), we conclude that the n-th cohomology is generated by the subvarieties of codimension n subjected to the relation that is exactly rational equivalence. This shows that $CH^n(X) \cong H^n(X, \mathscr{K}_n^M)$.

□

7.2.2 Milnor-Witt $\mathscr{K}$-sheaves and the Refined Chow-Witt groups

Our goal now is to introduce a new $\mathscr{K}$ sheaf, which will have similar properties to $\mathscr{K}^M$, but its cohomology will compute not the usual Chow group, but a "refined" Chow group, in which the coefficients are not integers, but quadratic forms, i.e., elements of the ring $GW(k)$. This will allow us to consider an intersection theory that takes the field into account. We follow the references [43] and [52, Chapter 2].

Definition 7.13. *For a ring R, the Milnor-Witt algebra $K_*^{MW}(R)$ is the algebra generated by elements $[u]$, for $u \in R^\times$, in degree 1, and a new symbol η in degree -1. This generator satisfies the relations:*

1. $[u]\eta = \eta[u]$, *for any* $u \in R^\times$,
2. $[u][1-u] = 0$, *for any* u, $1-u$ *in* $R^\times$,
3. $[uv] = [u]+[v]+\eta[u][v]$, *for any* $u, v \in R^\times$,
4. $\eta h = 0$, *where* $h = 2+\eta[-1]$.

Remark 7.14. *Relation (2) given in the definition above guarantees that the quotient $K_*^{MW}(R)/(\eta)$ is simply $K_*^M(R)$.*

Lemma 7.15. *We have that* $[1] = 0$ *and, as in Theorem 7.7, that*

1. $[u][-u] = 0$
2. $[u]^2 = u[-1]$ *and*
3. $[u^n] = n_\epsilon[u]$, *where*

$$n_\epsilon = \sum_{i=0}^{n-1} 1 + \eta[(-1)^i] = \begin{cases} \frac{n}{2}h, & \text{if } n \text{ is even} \\ 1 + \frac{n-1}{2}h, & \text{if } n \text{ is odd} \end{cases}$$

Proof. To show that $[1] = 0$, consider the two equations obtained by applying relation 3 in Definition 7.13.

$$[1] = [-1] + [-1] + \eta[-1]^2 = [-1]h$$

$$[1] = [1] + [1] + \eta[1]^2 \implies 0 = [1] + \eta[1]^2$$

By combining the two equations and applying relations 1 and 4 of the same definition:

$$0 = [1] + \eta[-1]h[-1] = [1] + [-1]\eta h[-1] = [1].$$

The rest of the proof is analogous to Theorem 7.7. We use again that $(1-u^{-1})(-u) = 1-u$. This implies that:

$$[1-u] = [1-u^{-1}] + [-u] + \eta[1-u^{-1}][-u] \tag{7.12}$$

Noticing that

$$0 = [1] = [u] + [u^{-1}] + \eta[u][u^{-1}] \implies [u] = (-1-\eta[u])[u^{-1}] = \alpha[u^{-1}]$$

and multiplying (7.12) by $[u]$ and using relations 1 and 3 in the definition 7.13 again, we have:

$$0 = [u][1-u] = \alpha[u^{-1}][1-u^{-1}] + [u][-u] + \alpha\eta[u^{-1}][1-u^{-1}][-u] = [u][-u]$$

The second assertion is again an easy consequence:

$$[u]^2 = [u]([-u] + [-1] + \eta[-u][-1]) = [u][-u] + [u][-1] + \eta[u][-u][-1] = [u][-1]$$

Finally, for the last one, we just use induction and the formula 3 of the definition 7.13

$$[u^n] = [u^{n-1}] + [u] + \eta[u^{n-1}][u] = (n-1)_\epsilon[u] + [u] + \eta(n-1)_\epsilon[u]^2 =$$

$$= [u]\left((n-1)_\epsilon + 1 + (n-1)_\epsilon\eta[-1]\right).$$

Now, if $n-1$ is even, i.e., n is odd, we get:

$$[u]^n = [u]\left(\frac{n-1}{2}h + 1 + 0\right) = n_\epsilon[u],$$

and, if $n-1$ is odd, i.e, n is even:

$$[u]^n = [u]\left(1 + \frac{n-2}{2}h + 1 + \eta[-1]\right) = [u]\left(\frac{n-2}{2}h + h\right) = [u]\frac{n}{2}h = n_\epsilon[u].$$

□

Proposition 7.16. *If F is a field, we have $K_0^{MW}(F) \cong GW(F)$.*

Proof. Define a map $\phi : GW(F) \to K_0^{MW}(F)$ on the generators by the equation

$$\phi(\langle u \rangle) = 1 + \eta[u],$$

and extend it by linearity.

1. ϕ is well defined. If $u = a^2$, we have:

$$1+\eta[a^2] = 1+\eta(2[a]+\eta[a]^2) = 1+\eta(2[a]+\eta[a][-1]) = 1+[a]\eta h = 1.$$

2. ϕ is a morphism, that is, it respects the multiplication:

$$\phi(\langle uv\rangle) = 1+\eta[uv] = 1+\eta([u]+[v]+\eta[u][v]) = (1+\eta[u])(1+\eta[v]).$$

3. ϕ is an isomorphism since it has an inverse given by sending $\eta[u]$ to $\langle u\rangle - \langle 1\rangle$.

□

Remark 7.17. *The image of the elements n_ϵ by ϕ will be denoted by the same n_ϵ in $GW(k)$.*

$$n_\epsilon = \sum_{i=0}^{n-1} \langle(-1)^i\rangle$$

Lemma 7.18. *Let R be a DVR. Let F be its field of fractions and let k be its field of residues and ϕ be the quotient map $R \to k$. Then, for each generator t of the maximal ideal of R, there exists a morphism ∂_t from $K_n^{MW}F$ to $K_{n-1}^{MW}k$ that commutes with η and satisfies:*

1. $\partial_t([u_1]\dots[u_n]) = 0$ if the u_i are all units on R;

2. $\partial_t([t][u_2]\dots[u_n]) = [\phi(u_2)]\dots[\phi(u_n)]$

Proof. We use the same idea as in Lemma 7.8. First, we define the map from $K_1^{MW}(F) \to K_0^{MW}(k) \cong GW(k)$. For $\alpha = ut^r \in F^\times$, we define:

$$\partial_t([ut^r]) = r_\epsilon \left\langle \phi(u)\right\rangle \in GW(k).$$

Notice that if we write

$$[ut^r] = [u]+[t^r]+\eta[u][t^r] = [u]+r_\epsilon[t]+\eta[u]r_\epsilon[t],$$

we see that the definition above agrees with properties one and two.

$$\partial_t[u]+\partial_t(r_\epsilon[t])+\eta\partial_t(r_\epsilon[t][u]) = r_\epsilon\left\langle\phi(u)\right\rangle$$

For r odd, for example, this implies:

$$\partial_t[u]+\partial_t\left([t]+\frac{r-1}{2}h[t]\right)+\eta\partial_t\left([t][u]+\frac{r-1}{2}h[t][u]\right) = 1+\eta[\phi(u)]+\frac{r-1}{2}h$$

$$\partial_t[u]+\partial_t([t])+\frac{r-1}{2}\partial_t\left(2[t]+\eta[t],[-1]\right)+\eta\partial_t([t][u]) = 1+\eta[\phi(u)]+\frac{r-1}{2}h,$$

from which, we conclude that $\partial_t([u]) = 0$, $\partial_t([t][u]) = [\phi(u)]$ and that $\partial_t(r_\epsilon[t]) = r_\epsilon$.

To show existence, we can show that any element can be written (up to η) as a sum of elements of the form $[t][u_1]\dots[u_n]$ or $[u_1]\dots[u_n]$. Indeed, this is done by writing a general element

$$\alpha = [u_1t^{r_1}]\dots[u_nt^{r_n}],$$

and using that

$$[ut^r] = [u]+[t^r]+\eta[u][t^r] = [u]+r_\epsilon[t]+\eta[u]r_\epsilon[t].$$

.

Finally, as the representation of an element is unique once t is fixed, the map is well defined by writing α as above. □

Remark 7.19. *Notice that the map above depends on the choice of t, differently from the one in 7.8. Also, another proof for the lemma above can be found in [52].*

We can now move back to geometry. Let X be a smooth variety over k. Following the same idea of Definition 7.10, we can define Milnor-Witt sheaves, denoted by $\mathscr{K}_n^{MW}$. Although the maps depend on the choice of a local parameter at each point, the cohomology (i.e., kernels and images) will not depend on this choice (again, for details, we refer to [52]). In this case, though, we are also interested in twisting our sheaves by line bundles L. To do this, notice that $\mathscr{K}_n^{MW}$ can be regarded as a $\mathbb{Z}[\mathscr{O}_X^\times]$ module, by considering the inclusion of $\mathscr{O}_{X,x}^\times$ on the stalks $\mathscr{K}_0^{MW}$. On the other hand, any line bundle L has an associated sheaf $L^\times$: the sheaf of non-vanishing sections, which is also an $\mathbb{Z}[\mathscr{O}_X^\times]$-module.

Definition 7.20. *Let X be smooth and let L be a line bundle. We define the twisted Milnor-Witt sheaf, denoted by $\mathscr{K}^{MW}(L)$, to be:*

$$\mathscr{K}^{MW} \otimes_{\mathbb{Z}[\mathscr{O}_X^\times]} L$$

The line bundles can be considered up to squares, that is, if $L \cong M^{\otimes 2}$, the twisting is trivial (since the squares of $\mathscr{O}_X^\times$ act trivially).

By analogy with the Theorem 7.11, we can define (twisted) Chow-Witt groups by taking the cohomology of the Milnor-Witt $\mathscr{K}$ sheaves.

Definition 7.21. *Let X be smooth over k and let L be a line bundle. The Chow-Witt groups twisted by L are defined as*

$$\widetilde{CH}^n(X,L) = H^n(X,\mathscr{K}_n^{MW}(L))$$

Remark 7.22. *By using the same argument as in the proof of Theorem 7.11, we can regard $\widetilde{CH}^n(X,L)$ as cycles with coefficients of quadratic forms over L.*

We have pullbacks and pushforwards for Chow-Witt groups, but we have to be careful about the line bundles. This new technical complication is a result of considering "orientations".

Proposition 7.23. *For $f: X \to Y$ of relative dimension d, we can define:*

- *A pullback* $f^*: \widetilde{CH}^n(Y,L) \to \widetilde{CH}^n(X,f^*L)$
- *A pushforward for f proper* $f_*: \widetilde{CH}^n(X,\omega_X \otimes f^*\omega_Y \otimes f^*L) \to \widetilde{CH}^{n-d}(Y,L)$

Proof. The proof follows from the constructions for classical Chow groups as the Chow Witt group can be seen as generated by algebraic cycles. See Remark 7.22. □

Remark 7.24. *The result above gives us a map $\int_X : \widetilde{CH}^n(X,\omega_X) \to GW(k)$, given by push-forwarding any class to the point $* = \mathrm{Spec}(k)$. It also defines the Euler class of a vector bundle, $\varepsilon(V) \in \widetilde{CH}^r(X,\det^{-1}(V))$, via pushfowarding the class 1_X by a section and then pulling it back. Notice that we can only compute the integral of an Euler class (that is, count the zeros of a generic section), if the determinant of the bundle is, up to squares, given by ω_X. This gives a notion of (relatively) orientable vector bundles. For the tangent bundle TX, we have that $\det^{-1}(TX) = \det^{-1}((\Omega_X^1)^{-1}) = \omega_X$. This implies that we can compute $\int_X \varepsilon(TX)$, for any X.*

Definition 7.25. *The Euler characteristic of a smooth and projective variety X can be defined as*

$$\chi^{\mathbb{A}^1}(X) := \int_X \varepsilon(TX)$$

If X is not projective, one can still define the Euler characteristic. We can consider the motivic stable homotopic category and use the fact that the infinite suspension spectrum of X is strongly dualizable and gives rise to an endomorphism of the sphere spectrum, which corresponds to an element of $GW(k)$. The relationship between these two definitions is explained in [42] and was first proved in [47]. The same idea can extend this map to a compactly supported Euler characteristic.

Chapter 8

Aritmetic Refinements of Donaldson-Thomas invariants

8.1 Arithmetic Donaldson-Thomas Invariants

After introducing $\mathbb{A}^1$ enumerative geometry and defining an $\mathbb{A}^1$ version of the Euler characteristic, we can consider the map defined on $K_0(Var(k))$ that takes any variety and evaluates its compactly supported Euler characteristic. Even though we have not defined this for non-projective varieties, we only need to consider projective varieties (see (6.1)). This was already pointed out in [6] and they stated the following:

Proposition 8.1 (cf. [6] Thm. 1.13)**.** *Let k be a field with $char(k) = 0$. Then the compactly supported $\mathbb{A}^1$-Euler characteristic is well defined and the following map:*

$$\chi_c^{\mathbb{A}^1} : K_0(Var(k)) \to GW(k)$$

is a homomorphism of rings.

The proof of the above proposition simply follows from the usual properties of Euler characteristics, which are compatible with the relations in $K_0(Var(k))$. The morphism from Proposition 8.1 can be extended to the localization of $K_0(Var(k))$ in the same way as the topological Euler characteristics (see Section 6.1) after adjoining a square root of $\langle -1 \rangle$ to $GW(k)$.

$$\chi_c^{\mathbb{A}^1} : \mathcal{M}_k \to GW(k)(\alpha) \tag{8.1}$$

where α is such that $\alpha^2 = \chi_c^{\mathbb{A}^1}(\mathbb{L}) = \langle -1 \rangle$

The morphism above allows us to get a "numerical" version of DT invariants over any field, given that the motivic version defined in Definition 6.10 is in $\mathcal{M}_k$ (seen as a subring of $\mathcal{M}_k^{\hat{\mu}}$). We call these invariants *arithmetic DT invariants*. One can also consider an equivariant version of the $\mathbb{A}^1$-Euler characteristic in order to define arithmetic invariants even when the virtual classes are not in $\mathcal{M}_k$.

An interesting question is whether there is a direct definition of such invariants that is not related to the Grothendieck ring of varieties but defined directly using the tools of $\mathbb{A}^1$-homotopy theory.

8.2 Local $\mathbb{A}^1$-degree and EKL Classes

We now discuss some aspects of the Eisenbud-Khimshiashvili-Levine (EKL) classes, which, as we stated in the introduction and section 6.3, are an important ingredient for possible physical interpretations and might be key to finding a relationship between the motivic nearby cycles and the $\mathbb{A}^1$-version of the Milnor number.

In the papers [22] and [38], the idea was to find a way of computing the local topological degree of $P : \mathbb{R}^n \to \mathbb{R}^n$ from algebraic information in the local ring of P at 0. They showed that there is a quadratic form (the EKL form) defined on the local algebra whose signature corresponds to the local degree. The same problem was considered for holomorphic maps, and the rank of the EKL form ended up being equal to the local degree.

The rank and signature appearing above are already a hint that, over an arbitrary field k, the class of the EKL form (EKL class) in $GW(k)$ should be equal to the refinement of the local topological degree introduced in Definition 7.3. Using such refinement, J. Kass and K. Wickelgren showed that the class of the EKL form in $GW(k)$ correspond to the local $\mathbb{A}^1$-degree [35]. We now define the EKL form.

Definition 8.2. *Consider a morphism $P : \mathbb{A}^n \to \mathbb{A}^n$ with $P(0) = 0$. Assume that this zero is isolated. We can write P as $(P_1,\dots,P_n)$, with $P_i = \sum_{j=1}^n a_{ij}x_j$. Let $A = k[x_1,\dots,x_n]_{(x_1,\dots,x_m)}/(P)$ be the local algebra of P at 0. Define $E = det(a_{ij})$.*

Let $\phi : A \to k$ be any k-linear map and define the bilinear form $\beta_\phi(p,q) = \phi(pq)$. The Eisenbud-Khimshiashvili-Levine (EKL) class of P is the class of $\beta_\phi \in GW(k)$ for any ϕ with $\phi(E) = 1$. It is denoted by $w_0(P)$.

Remark 8.3. *E basically carries the same information as the Jacobian determinant $\det J$ of P. Specifically, $\det J = \dim_k A \cdot E$. In particular, in characteristic zero, one can consider J instead of E.*

Remark 8.4. *If P has a simple zero at the origin, then $w_0(f)$ is simply the class $\langle E\rangle = \langle \det J\rangle$. This follows from the fact that in this case $A \cong k$. In particular, it corresponds to the local degree in this case, since P having a simple zero implies P is étale.*

EKL classes are related to critical loci and Milnor fibres for isolated singularities. For $f : \mathbb{A}^n \to k$, the derivative of f gives us a map $P := df : \mathbb{A}^n \to \mathbb{A}^n$ as we considered above. Then, the EKL class $w_0(P)$ refines the Milnor number of the singularity. Indeed, over $\mathbb{C}$, the Milnor number is the vector space dimension of the quotient A, which is, by definition, the rank of the EKL quadratic form. This refinement was introduced in [35, Section 6].

Over $\mathbb{C}$, the Milnor number of f is closely related to the topology of the Milnor fibre, which is homotopic to a bouquet of μ spheres S^n, where μ is the Milnor number of f. This gives us the classical formula:

$$\chi(F) = 1 + (-1)^{n-1}\mu(f),$$

where F is the Milnor fibre of f. We suspect that this formula can be generalized.

Example 8.5. *In the case considered in Definition 6.9, we had $f : \mathbb{A}^2 \to k$ given by $x^2 - y^2$. Its derivative is given by*

$$P := df : \mathbb{A}^2 \to \mathbb{A}^2$$
$$(x,y) \mapsto (2x, -2y).$$

The Jacobian matrix, in this context, is simply the Hessian of f. This gives us:

$$J(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix},$$

for any (x,y).

Now, the $\mathbb{A}^1$-Milnor number can be easily computed by the consideration of Remark 8.4 and Equation 8.2.

$$\mu^{\mathbb{A}^1} = \langle \det J(0)\rangle = \langle -4\rangle = \langle -1\rangle \in GW(k).$$

Finally,

$$\chi_c^{\mathbb{A}^1}(S_{f,0}) = \chi_c^{\mathbb{A}^1}(1-\mathbb{L}) = \langle 1\rangle - \langle -1\rangle = \langle 1\rangle + (-\langle 1\rangle)^{2-1}\cdot\langle -1\rangle = \langle 1\rangle + (-\langle 1\rangle)^{n-1}\mu^{\mathbb{A}^1}(f).$$

Of course, in general, one has to consider an equivariant version of the $\mathbb{A}^1$-Euler characteristic, which takes into account the action of $\hat{\mu}$, since the classes $S_{f,x}$ are in $\mathcal{M}_k^{\hat{\mu}}$. Many interesting questions can be asked regarding this topic: what is the general relationship between the $\mathbb{A}^1$-Milnor number of a function f and the $\mathbb{A}^1$-Euler characteristic of the motivic class S_f? Is there anything that can be said for non-isolated singularities?

8.3 Comparison with Related Work

As we mentioned in the introduction, Levine [44] has given an alternative definition of "arithmetic DT invariants", independently of the motivic invariants. This construction relies on the definition of DT invariants for general Deligne-Mumford stacks due to Behrend and Fantechi [9]. The idea is to construct a motivic version of the intrinsic normal cone, called the "intrinsic stable normal cone". For a scheme Z over B, this cone is an object in the motivic stable homotopy category from $\mathbb{A}^1$-homotopy theory and, given any perfect obstruction theory $\phi : E_\bullet \to \mathbb{L}_{Z/B}$, it can be used to construct a class $[Z,\phi]^{vir} \in \widetilde{CH}_r(Z;\det E_\bullet)$. Now, if $\det E$ is isomorphic to ω_Z up to squares, integration gives us an invariant in $GW(k)$, as we described in 7.24. The choice of an obstruction theory such that E is relatively orientable is therefore essential for this theory.

In the case we considered above, i.e., Z given as a critical locus of a function f on a smooth variety M, we have that Z is given by zeros of a section of the cotangent bundle. In this case, there is a canonical obstruction theory induced by the surjective morphism $df : \mathcal{T}_M \to \mathcal{I}_Z \to 0$, where $\mathcal{I}_Z$ is the ideal corresponding to Z. After tensoring by $\mathcal{O}_Z$, we see that df induces a map $F : \mathcal{T}_M \otimes \mathcal{O}_Z \to \mathcal{I}_Z/\mathcal{I}_Z^2$. Then, by considering the composition with the derivative in the first term of the complex, we get the obstruction theory

$$\phi = (F, \mathrm{Id}) : \left(\mathcal{T}_M \otimes \mathcal{O}_Z \xrightarrow{\partial_\phi} \Omega_{M/B} \otimes \mathcal{O}_Z\right) \to \left(\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} \Omega_{M/B} \otimes \mathcal{O}_Z\right),$$

which induces a virtual class $[Z]^{vir}$ in the Chow-Witt group of Z twisted by $\det(T_M)\otimes O_Z$, which is clearly orientable.

A more detailed comparison can be made based on the works of Azouri [7] and Levine-Pepin-Lehalleur-Srinivas [46]. Specifically, the conductor formulas for the nearby fiber functor given in these works show that our invariants would agree with the ones in [44] over $\mathbb{R}$, but not necessarily in general. It is also worth mentioning that such formulas also give partial answers to the questions at the end of the last section, with the formula $\langle 1\rangle + (-\langle 1\rangle)^{n-1}\mu^{\mathbb{A}^1}(f)$ for the Euler characteristic of the motivic Milnor fibre being correct in for the case of mappings $\mathbb{A}^n \to \mathbb{A}^1$ given by quasi-homogeneous singularities. Finally, we point out that adjoining α to $GW(k)$ in (8.1) is not related to orientation issues, but simply due to the appearance of the class $\mathbb{L}^{\frac{1}{2}}$ in $K_0(Var(k))$ needed to make up for the sign differences (see remark 6.11).

8.4 Examples from Physics and Computations

In this last section, we show how, starting from previous computations, we can get results on $GW(k)$. Even though these have still some conjectural aspects, they show how the arithmetic DT invariants should behave. We are especially interested in the comparison between our arithmetic invariants (and also the motivic invariants) with real invariants for the case of degree zero DT invariants of $\mathbb{A}^3$, which, as far as we are concerned, was not pointed out before.

We also consider the example of Gopakumar-Vafa invariants, generalizing a result of Liu and Ruan, even though there is no real version of GV invariants, pointing out these results might guide future research.

8.4.1 The Hilbert Scheme of Points of the Affine Space as a Critical Locus

We start by computing degree 0 DT invariants of $\mathbb{A}^3$. The moduli space we have to consider, therefore, is simply the Hilbert Scheme of n points of $\mathbb{A}^3$ (see Section 6.2). We recall how to realize the Hilbert scheme of n points of $\mathbb{A}^3$ as a critical locus of a function. Fix the notation $\mathrm{Hilb}^n(\mathbb{A}^r)$ for the Hilbert scheme of n points of $\mathbb{A}^r$ and

$$\mathbb{A}^k = \mathrm{Spec}\, k[x_1,\dots,x_r] = \mathrm{Spec}\, k[x].$$

The following considerations can be found in the notes by Nakajima [60].

In the affine space, a subscheme $B \subset \mathbb{A}^r$ of dimension 0 and degree n correspond to a quotient of $k[x]$ of dimension n. Therefore, if we fix a vector space V of dimension n, the structure we need to add in order to get a quotient of $k[x]$ consists of an action of $k[x]$ on V and an element $v \in V$ (that will correspond to 1) that generates the whole space under the action. Such action is the choice of r elements of $\mathrm{Hom}(V,V)$ which commute. This allows us to consider the set:

$$B(V) = \{(A_1,\dots,A_r,v) \in \mathrm{Hom}^r(V,V)\times V \mid [A_i,A_j]=0,\ v \text{ generates } V \text{ under the action}\}.$$

To get the Hilbert Space, where each element corresponds to a different quotient of $k[x]$, from $B(V)$, we need to mod out by the action of $GL(V)$ (by conjugation on the A_i).

Proposition 8.6. *Let $V \cong k^n$ be an n-dimensional k-vector space. The Hilbert scheme of points of $\mathbb{A}^r$ can be represented as*

$$\mathrm{Hilb}^n(\mathbb{A}^r) \cong \left\{ (A_1,\dots,A_r,v) \,\middle|\, \begin{array}{l} \text{(i) } [A_i,A_j]=0 \textit{ for all } i,j \\ \text{(ii) } v \textit{ generates } V \textit{ under the action of the } A_j \end{array} \right\} \Big/ \mathrm{GL}_n(r), \tag{8.2}$$

Proof. Each point on $\mathrm{Hilb}^n(\mathbb{A}^r)$ can be represented by an ideal $I \subset k[x]$. Take any isomorphism $\frac{k[x]}{I} \cong V$. Let A_i be given by multiplication by x_i and let $v = \bar{1} \in \frac{k[x]}{I}$. With such choices, *(i)* and *(ii)* are obvious. Notice that this map does not depend on the choice of the isomorphism since any two isomorphisms are related by conjugation by elements $GL_n(r)$ on V.

On the other hand, given an element $(A_1,\dots,A_r,v)$ satisfying *(i)* and *(ii)*, we can define a k-algebra structure on V via the map from $k[x] \to V$ given by sending x_i to $A_i v$ and 1 to v. This map is well defined by *(i)* and it is surjective by *(ii)*. By taking the kernel, we get an ideal $I \subset k[x]$ which represents an element of the Hilbert scheme. These considerations imply that these two maps are inverses and therefore define an isomorphism. □

Let us now restrict ourselves to the case $r = 3$. In this situation the above result from Proposition 8.6 allows us to write $\mathrm{Hilb}^n(\mathbb{A}^3)$ as the critical locus of a regular function on a smooth space. This puts us in the situation that we have already studied in Section 8.1 of this chapter and in Section 6.3 of Chapter 6.

Proposition 8.7. *Fix a vector space $V \cong k^n$ of dimension n. Consider the quotient*

$$M_n = \frac{\{(A,B,C,v) \mid v \textit{ generates } V \textit{ under the action of the algebra generated by } A, B \textit{ and } C\}}{\mathrm{GL}_n(V)},$$

and the function

$$f_n : M_n \to k$$
$$(A,B,C,v) \mapsto \mathrm{Tr}([A,B]C)$$

Then M_n is a smooth variety, f_n is a regular map and $\mathrm{Hilb}^n(\mathbb{A}^3) = \{df_n = 0\} \subset M$

Proof. Most of our argument comes from [11] and [65]. To see that M_n is an algebraic variety, it suffices to notice that M_n is a quotient of an open subset of a vector space by a free action. Indeed, the condition of the v generating the whole V under the action is an open condition. To see that the action is free, we have:

$$g(A,B,C,v) = (gAg^{-1}, gBg^{-1}, gCg^{-1}, gv) = (A,B,C,v) \implies$$

$$\implies W = \ker(g - id) \text{ is stable under the action of } A, B \text{ and } C \text{ but contains } v$$

By definition, v generates V under the action and so $W = V$. This shows $g = id$. Now we use a construction from GIT. We consider the action on $\mathrm{Hom}(V,V)^3 \times V \times \mathbb{C}$ given by

$$g(A,B,C,v,z) = (gAg^{-1}, gBg^{-1}, gCg^{-1}, gv, det(g)^{-1}z)$$

.

The semistable points of this action are given exactly by points whose closures of their orbits are disjoint of $\mathrm{Hom}(V,V)^3 \times V \times 0$. We see that, if v generates V under the action, this condition is satisfied. GIT, then, gives us that the quotient is a smooth variety.

Now, by Proposition 8.6, it is enough to show that the condition $df_n = 0$ corresponds to commutativity of the A_i. Indeed, we have:

$$\mathrm{Tr}([A,B]C) = \sum_i \sum_k \sum_j (a_{ij}b_{jk} - b_{ij}a_{jk})c_{ki}.$$

Therefore, if the derivatives with respect to each entry of C are all zero, we get that $[A,B] = 0$. As $\mathrm{Tr}([A,B]C) = \mathrm{Tr}(A[B,C]) = \mathrm{Tr}(B[C,A])$, the vanishing of the other derivatives implies that the other pairs of matrices commute. □

8.4.2 The Virtual Classes of the Hilbert Scheme of Points

To compute the virtual classes over $\mathbb{C}$, the authors of [11] used Proposition 6.13 and the fact that there is a natural toric action on M which descends given by

$$(t_1, t_2, t_3) \cdot (A,B,C,v) \mapsto (t_1A, t_2B, t_3C, t_1t_2t_3v)$$

which satisfies all the hypotheses. For details, see [11, Lemma 2.4].

This implies that the virtual class of $\mathrm{Hilb}^n(\mathbb{C}^3)$ can be computed by the difference $[f_n^{-1}(1)] - [f_n^{-1}(0)]$ as in Theorem 6.13. This difference was computed in [11] to correspond to a generating series

$$Z_{\mathbb{C}^3}(t) = \sum_{n=0}^{\infty} [\mathrm{Hilb}(\mathbb{C}^3)]_{vir} t^n = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} (1 - \mathbb{L}^{k+2-m/2} t^m)^{-1}. \tag{8.3}$$

This computation (cf. Theorem 3.7 in [11]) relies on the motivic classes of Grassmanians, general linear groups, and of the variety of commuting matrices, which are the same over any field. The only difficulty in generalizing this computation would be to prove that the virtual class is given by the difference of the fibers. Sample targets for generalization include examples studied by Choi-Katz-Klemm in [16]. We next explain how to obtain the class of the variety of commuting matrices over any field, which is something that doesn't appear in the paper [11]. There, they refer to the computation of the number of points in this variety over finite fields ([25]). We explain how to adapt the argument.

Lemma 8.8 (cf. Prop. 2.1 of [11]). *Let V be a vector space of dimension n and let $C_n \subset Hom(V,V)^2$ be the variety of pairs of commuting linear transformations on V. Define the class*

$$\tilde{c}_n = \frac{[C_n]}{[GL_n]} \in K_0(Var(k))[(1-\mathbb{L}^n)^{-1} : n \geq 1].$$

We have

$$C(t) = \sum_{n=0}^{\infty} \tilde{c}_n t^n = \prod_{m=1}^{\infty} \prod_{j=0}^{\infty} (1-\mathbb{L}^{1-j} t^m)^{-1}$$

Proof. As stated in [11], the proof follows from the count of the number of pairs of commuting matrices over finite fields by Feit and Fine [25] that we adapt here. For each linear transformation $A \in Hom(V,V)$, we have a decomposition

$$V = K_A \oplus I_A,$$

where where $K_A = \ker A^n$ and $I_A = \operatorname{Im} A^n$ (recall $n = \dim V$). Notice that A and B commute if and only if $B(K_A) \subset K_A$, $B(I_A) \subset I_A$ and $A|_{K_A}$ commutes with $B|_{K_A}$ and $A|_{I_A}$ commutes with $B|_{I_A}$.

Now, fix K and I in V with $\dim K = s$, $\dim I = t$ and $K \oplus I = V$. Define $C_n^{K,I} = \{(A,B) \in Hom(V,V)^2 \mid K_A = K \text{ and } I_A = I\}$. By the consideration above, we have the decomposition:

$$C_n^{K,I} = C_K^{\text{nil}} \times C_I^{\text{inv}}$$

where

$$C_K^{\text{nil}} := \{(A,B) \in \operatorname{Hom}(K,K)^2 \mid A \text{ is nilpotent}\} \text{ and } C_I^{\text{nil}} := \{(A,B) \in \operatorname{Hom}(I,I)^2 \mid A \text{ is invertible}\}.$$

Indeed, if A and B commutes and $K = K_A$ and $I = I_A$, we have that the restrictions of A and B to K and I are well defined and, by definition, $A|_K$ is nilpotent and $A|_I$ is invertible (since A^n is invertible when restricted). On the other hand, if $(A',B') \in C_K^{\text{nil}}$ and $(A'',B'') \in C_I^{\text{inv}}$, $A = A' \oplus A$ satisfy $K_A = K$ and $I_A = I$ (by definition) and commutes with $B = B' \oplus B''$. Now, noticing that choosing K and I is the same thing as choosing a basis of V (element in GL_n) and factoring by choices of bases of K and I (elements of GL_s and of GL_t), we have, after taking classes in $K_0(Var(k))$

$$C_n = \bigcup_{s+t=n} \frac{GL_n}{GL_s \times GL_t} C_s^{\text{nil}} \times C_t^{\text{inv}}$$

(as the spaces only depend on the dimension, we switched K and I by the dimensions). It is then enough to compute C_t^{inv} and C_s^{nil}.

To compute C_s^{nil}, fix a nilpotent matrix A. The similarity classes of A matrices can be identified with partitions of the dimension, i.e., ways of writing $s = r_1 + \ldots r_k$. This is done by choosing a good triangular representation (see [3]). The matrices that commute with A form a linear space of dimension $M(\pi)$, where $M = \sum_{i,j} \min(r_i, r_j)$ for each partition π of s. This can be computed by choosing a basis that triangularizes A. As we only want one for each class, we have to take the quotient by the non-singular matrices that commute with A. The reader can check the details in [26]. We get:

$$[C_s^{\text{nil}}] = [GL_s] \sum_{\pi(s)} \frac{[\mathbb{L}^{b_i^2}]}{[GL_{b_i}]}$$

For C_t^{inv}, notice that the space β of matrices similar to a matrix B is simply the quotient of GL_t by the subgroup $C^{\text{inv},\beta}$ of GL_t formed by matrices that commute with B (since $gB = Bg \implies gBg^{-1} = B$, which means that g does not change the similarity class). Therefore, we have $[C^{\text{inv},\beta}][\beta] = [GL_t]$ and if S_t is the space of similarity classes of $t \times t$ matrices, i.e., considering all βs:

$$[C_t^{\text{inv}}] = [S_t][C^{\text{inv},\beta}][\beta] = [S_t][GL_t]$$

It suffices, then, to compute $[S_t]$. Each element in $[S_t]$ is a matrix given by a rational canonical form, for which corresponds a set of t monic polynomials g_i with $g_i | g_{i+1}$ and with sum of degrees equal to t. Setting $h_i = \frac{g_{i+1}}{g_i}$, the degrees of h_i satisfy $\sum_{i=1}^{t} i \deg h_i = t$. This sum corresponds to a partition of t with b_i as the frequency of i. For each partition π, the space of polynomials that correspond to such a partition is simply the vector space given by the sum of the degrees. Letting $k(\pi)$ be the sum of the frequencies for the partition π:

$$[S_t] = \sum_{\pi \in \pi(t)} \mathbb{L}^{k(\pi)}$$

To finish the proof, one has to use many identities of partition functions that we omit here.

□

Remark 8.9. *The fact that the computation relies on the number of pairs of commuting matrices, whose class is computed from partitions, makes the fact that you get the MacMahon function*

$$M(t) = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)^n}, \tag{8.4}$$

which is the generating function for plane partitions, more plausible. In the end, the geometry "hides" identities of partition functions.

Even though we do not have a generalization for Proposition 6.13 and therefore do not have a partition function defined over any field, the expression over $\mathbb{C}$ is well defined over any field (since the only variety appearing is $\mathbb{L}$). Therefore, to test our definition of arithmetic DT invariants 8.1, we can use the morphism in equation 8.1, to get arithmetic DT invariants and compare with the real invariants. It is interesting that the motivic invariants (even though only computed over $\mathbb{C}$ already contain information about the real geometry).

Proposition 8.10. *The DT invariants for $\mathbb{A}^3$ can be refined over $GW(k)$ by the generating series:*

$$\prod_{n=1}^{\infty} (\langle 1\rangle - (\alpha t)^{2n-1})^{-1} \prod_{n=1}^{\infty} (\langle 1\rangle - (\alpha t)^n H + \langle -1\rangle(\alpha t)^{2n})^{-\lfloor \frac{n}{2} \rfloor} \tag{8.5}$$

Proof. Applying the morphism, we first just send $\mathbb{L}$ to $\langle -1\rangle$ and $\mathbb{L}^{\frac{1}{2}}$ to α.

$$\prod_{m=1}^{\infty} \prod_{k=0}^{m-1} (\langle 1\rangle - \langle -1\rangle^{k+2} \alpha^{-m} t^m)^{-1}$$

Notice that

$$(\langle 1\rangle - \langle -1\rangle \alpha^m t^m)(\langle 1\rangle - \alpha^m t^m) = (\langle 1\rangle - \alpha^m t^m H + \langle -1\rangle \alpha^{2m} t^{2m})$$

where H is the hyperbolic form given by $\langle 1\rangle + \langle -1\rangle$.

In the case m is even, the product above will appear exactly $\frac{m}{2}$ times. For m odd, it appears $\frac{m-1}{2} = \lfloor \frac{m}{2} \rfloor$ times and we get an extra factor of the form $(\langle 1\rangle - \alpha^m t^m)^{-1}$.

$$\prod_{m=1}^{\infty} \prod_{k=0}^{m-1} (\langle 1\rangle - \langle -1\rangle^{k+2} \alpha^m t^m)^{-1} = \prod_{m \text{ odd}} (\langle 1\rangle - \alpha^m t^m)^{-1} \prod_{m=1}^{\infty} (\langle 1\rangle - (\alpha t)^m H + \langle -1\rangle (\alpha t)^{2m})^{-\lfloor \frac{m}{2} \rfloor}$$

By making $m = 2n-1$ in the first product and $m = n$ in the second, we get:

$$\prod_{n=1}^{\infty} (\langle 1\rangle - (\alpha t)^{2n-1})^{-1} \prod_{n=1}^{\infty} (\langle 1\rangle (\alpha t)^n H + \langle -1\rangle (\alpha t)^{2n})^{-\lfloor \frac{n}{2} \rfloor} \tag{8.6}$$

□

Remark 8.11. *The above refinement is compatible with previous results over $\mathbb{R}$ and $\mathbb{C}$.*

Taking $k=\mathbb{C}$, we have $GW(k)=\mathbb{Z}$ and $\alpha=-1$, which results in the classical MacMahon generating function for the number of plane partitions 8.4:

$$\prod_{n=1}^{\infty}(1-(-t)^{2n-1})^{-1}\prod_{n=1}^{\infty}(1-2(-t)^n+(-t)^{2n})^{-\lfloor\frac{n}{2}\rfloor}=$$
$$\prod_{n=1}^{\infty}(1-(-t)^{2n-1})^{-1}\prod_{n=1}^{\infty}(1-(-t)^n)^{-2\lfloor\frac{n}{2}\rfloor}=\prod_{n=1}^{\infty}(1-(-t)^n)^{-n}=M(-t)$$

For $k=\mathbb{R}$, after computing the signature morphism (i.e., sending $\langle 1\rangle$ to 1, $\langle -1\rangle$ to -1 and taking $\alpha=i$), we get the symmetric MacMahon function, which is the generating function for symmetrical plane partitions, which correspond to the real count as computed by Pasquetti-Krefl-Walcher in [41].

$$\prod_{n=1}^{\infty}(1-(-it)^{2n-1})^{-1}\prod_{n=1}^{\infty}(1-0(-it)^n+(-it)^{2n})^{-\lfloor\frac{n}{2}\rfloor}=$$
$$\prod_{n=1}^{\infty}(1-(-it)^{2n-1})^{-1}\prod_{n=1}^{\infty}(1-(-it)^{2n})^{-\lfloor\frac{n}{2}\rfloor}=M^{\mathrm{sym}}(-it)$$

8.4.3 Gopakumar-Vafa Invariants at the Castelnuovo Bound

Our interest now turns to the computation of Gopakumar-Vafa (GV) invariants for M a smooth quintic hypersurface in $\mathbb{P}^4$, through their relation to DT invariants. In this case, the moduli spaces of interest are $I_n(M,d)$, which correspond to the Hilbert scheme parameterizing subschemes of M with Hilbert polynomial given by $dt+n$, that is, curves of degree d and arithmetic genus $1-n$. The GV invariants n_g^d correspond to DT invariants $I_{1-g,d}$. Recent work by Liu-Ruan [49] and Alexandrov-Feyzbakhsh-Klemm-Pioline-Schimannek [4] has established the famous Castelnuovo bound for Gopakumar-Vafa invariants, which was predicted in physics,

$$n_g^d=I_{1-g,d}=0,\ \text{for any } d \text{ and } g \text{ with } g>\frac{d^2+5d+10}{10}=:B(d).$$

and led to a computation of the numbers $n_{B(d)}^d=I_{1-B(d),d}$. That is, the numbers n_g^d when the pair (g,d) is on the bound. Here, we write formulae for the motivic and arithmetic refinements of such numbers at the bound. This was done using the fact that, for $n=B(d)$, $\mathcal{M}_{n,d}$ is not only smooth but a projective bundle over a projective space (see [49, Prop. 6.2], where they prove it over $\mathbb{C}$).We believe that this is true over any k of characteristic zero.

Proposition 8.12. *Let $g=B(d)$. This implies that $B(d)$ is an integer, which means that d can be written as $d=5m$ for some m. Assuming that we can write the moduli space as a projective bundle as above, motivic and arithmetic refinements of the invariants $I_{n,d}$ for $n=1-g=1-B(d)$, are given by the formulae:*

$$[\mathcal{M}_{n,d}]_{vir}=\mathbb{L}^{\frac{N}{2}+2}\frac{(\mathbb{L}^{N+1}-1)(\mathbb{L}^5-1)}{(\mathbb{L}-1)^2}\in\mathcal{M}_k$$

and, applying the morphism $\chi^{\mathbb{A}^1}$:

$$\chi^{\mathbb{A}^1}\left([\mathcal{M}_{n,d}]_{vir}\right)=\begin{cases}\alpha\cdot\left(\frac{6+5N}{2}\langle 1\rangle+\frac{4+5N}{2}\langle -1\rangle\right), \textit{for } m=0,1 \bmod 4\\ \frac{5(N+1)}{2}H, \textit{for } m=2,3 \bmod 4\end{cases}$$

where $N=\binom{m+3}{3}-\binom{m-2}{3}-1$.

Proof. The base is $\mathbb{P}^4$ and the fibre is $\mathbb{P}^N$, where $N = \binom{m+3}{3} - \binom{m-2}{3} - 1$.

The first part of the result is simply given by using the fact that, for smooth varieties, the motivic virtual class is simply the class of the variety times $\mathbb{L}^{-\dim \mathcal{M}_{n,d}/2}$ (see Definition 6.10 and Example 6.12). To compute $[\mathcal{M}_{n,d}]$, we use that the class of a fibre bundle is the product of the fibre and the base.

Using that

$$[\mathbb{P}^r] = \mathbb{L}^r + \mathbb{L}^{r-1} + \dots \mathbb{L} + 1 = \frac{\mathbb{L}^{r+1}}{\mathbb{L}-1}$$

and that $\dim \mathcal{M}_{d,n} = N+4$, we get

$$[\mathcal{M}_{n,d}]_{vir} = \mathbb{L}^{-\frac{\dim \mathcal{M}_{n,d}}{2}}[\mathcal{M}_{n,d}] = \mathbb{L}^{\frac{N}{2}+2}[\mathbb{P}^N]\cdot[\mathbb{P}^4] = \mathbb{L}^{\frac{N}{2}+2}\frac{(\mathbb{L}^{N+1}-1)(\mathbb{L}^5-1)}{(\mathbb{L}-1)^2} \in \mathcal{M}_k.$$

Finally, applying the morphism, we only need to keep track of whether N is even or odd. If it is even, the $\mathbb{A}^1$-Euler characteristic of $\mathbb{P}^N$ is $\frac{N+1}{2}H$ and if it is odd it is given by $\frac{N+1}{2}\langle 1\rangle + \frac{N-1}{2}\langle -1\rangle$. $\square$

The formula above, especially the one in $GW(k)$, gives a prediction of what should be the $\mathbb{A}^1$-count of curves of higher genus on the quintic over any field. It would be interesting to check if this result can be reached with direct methods, without making use of motivic DT invariants.

Final Considerations

Although the work was divided into two quite different parts, they are similar in the sense that both of them concern the study of mathematical properties of Physical invariants. In the first, we have explained a possible way of interpreting modularity for Gromov-Witten invariants, especially for the open/real case. In the second part, we considered refined Donaldson-Thomas invariants and their relations with real invariants and with arithmetic: we were able find new relationships between different refinements clear and to give applications.

We would like to use this last section to expose some of the open problems that arose from the objects studied in this Thesis. For example, by considering a divisor family, we should be able to construct some kind of generalization of Jacobi forms: in the simple case of a divisor family given by two points inside an elliptic curve, Cao, Movasati, and Villaflor [15] recovered actual Jacobi forms, as we have mentioned in Chapter 5. Physically, an interesting example would be to consider the results of the paper [34] and try to recover their numbers from the Picard-Fuchs-type equations presented there.

Question 1. *Can we get a generalization of Jacobi forms considering the Gauss-Manin connection and the mixed Hodge structure for families of more general Calabi-Yau varieties with divisors?*

Another very important aspect to consider is the relationship that the modularity that we study here and in the GMCD program has with the literature.

Question 2. *How is the modularity found in GMCD related to other types of modularity found in the literature?*

For example, the classical Yau-Zaslow formula gives the generating series of the number of curves of genus g on a K3 surface of fixed homology class as quasi-modular forms. This was further extended, for example, in the work of G. Oberdieck [61]. Similarly, F. Greer [29] considered generating functions of GW invariants in cohomology rings which allowed some generalizations. Considering whether GMCD would give us similar results for these cases and also whether the generalizations constructed under GMCD would be in some sense related to the works above are problems that are worth studying.

Regarding the second part, we also have many interesting questions to consider. The first was already considered in 8.3 and is regarding how our arithmetic invariants relate to other works present in the literature. We are especially interested in the relations with Levine's construction in [44].

Question 3. *How is our construction related to previous work in the literature?*

Another interesting aspect for further study is to try to make the idea of applying Levine's localization formulas [45] to get a refined version of Theorem 6.13 and therefore be able to prove that our refinement for the degree zero invariant of $\mathbb{A}^3$ is correct. We pose this question in the following way:

Question 4. *Is there a way of using Levine's localization formula to show that the motivic Milnor fibre of the Hilbert scheme is the difference between the generic fibre and the zero fibre?*

We finish relating the two parts. There are still many open questions in both fields to consider and there are even relations between the two parts that might be very interesting. One example would be the question about the modularity of arithmetic invariants.

Question 5. *Is there a modularity for invariants valued at $GW(k)$? Is there an intrinsic way of considering this modularity?*

To be more specific regarding the last question, a possible line of research would be to construct some kind of $\mathbb{A}^1$-mirror symmetry for the quintic, at least. Recall that mirror quintic relates, on one side (the A side), Gromov-Witten invariants and, on the other side (the B side), computations with periods, which are defined in a purely algebraic setting (as we explained in the first part of this work). The idea, then, would be to relate the $\mathbb{A}^1$ counts of rational curves on the quintic to some class of $\mathbb{A}^1$-periods with values on $GW(k)$. Although the $\mathbb{A}^1$-counts are conjectured to be simply combinations of real and complex counts

$$n_d^{\mathbb{A}^1} = \frac{n_d^{\mathbb{C}} + n_d^{\mathbb{R}}}{2}\langle 1\rangle + \frac{n_d^{\mathbb{C}} - n_d^{\mathbb{R}}}{2}\langle -1\rangle$$

which are both in generating series

$$\sum_{d=0}^{\infty} n_d^{\mathbb{C}} d^3 \frac{q^d}{1-q^d}$$
$$\sum_{d \text{odd}} n_d^{\mathbb{R}} d^2 \frac{q^{d/2}}{1-q^d}$$

that can be written in terms of solutions of the vector fields from the first part of the thesis. However, it is not straightforward to find a suitable generating series for $n_d^{\mathbb{A}^1}$ (i.e., that satisfies some kind of vector field) because of the different exponents of d and q above. Even though, we think that it should be possible to construct a refined Gauss-Manin connection on a cohomology with coefficients on GW and, from that, compute new vector fields whose solutions should correspond to $n_d^{\mathbb{A}^1}$.

Question 6. *Is there a generating series for the $\mathbb{A}^1$-counts of rational curves on the quintic that satisfies differential equations of periods as in the classical case?*

We hope that this work was able to, at least, inspire mathematicians to pursue problems at the edge between Number Theory, Algebraic Geometry, and Physics and that discoveries regarding many of the interesting phenomena described here, from modularities that show up to the relationships between real and motivic invariants, will be done and clarify our understanding.

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